Lecture Notes, M261-004, Introduction to Vectors

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1 Definition

One of the most important concepts in math is the vector. Vaguely, we define a vector by a **direction** and a **magnitude.** Graphically, we represent a vector as a **directed line segment** from an **initial point** A to a **terminal point** B. The magnitude is given by the length of this segment. We say that two vectors are equal if they have the same magnitude and direction. Hence the same vector can be drawn from many different starting points. We usually denote a vector by a bold, lowercase letter, $\mathbf{v}$, or a letter with an arrow on top, $\vec{v}$.

Analytically, we usually express vectors in component form. We do this by considering the vector with the origin as its initial point and write the coordinates of its terminal point. For example, in 2D,

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

and in 3D,

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

Since the vector $\langle v_1, v_2 \rangle$ has 0 as its inital point and (v_1, v_2) as its terminal point, its magnitude is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

Similarly, a 3D vector in component form has magnitude

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

The only vector with 0 magnitude is the zero vector, with all components equal to 0.

Example 1. *Find the component form and magnitude of the vector with* $(4, 2, -1)$ *as its initial point and* $(-2, 3, 0)$ *as its terminal point.*

The component form of the vector is given by

$$\langle -2 - 4, 3 - 2, 0 - -1 \rangle = \langle -6, 1, 1 \rangle$$

Its magnitude is given by

$$\|v\| = \sqrt{36 + 1 + 1} = \sqrt{38}$$

Example 2 (From Textbook)**.** *A small cart is being pulled along a smooth horizontal floor with a 20-lb force* F *making a* $45°$ *angle with the floor. What is the horizontal component of the force?*

We have here a different way to represent a vector, but we still have a magnitude (20 lbs) and a direction (45°). The appropriate way to attack the problem is to graph the situation with the cart at the origin and the ground on the x-axis. Draw the force vector with magnitude 20 making a 45° angle with the x-axis. We now see that the x-component of the force is

$$20 \cos(45^\circ) = 14.14$$

The force vector is in fact

$$\langle 14.14, 14.14 \rangle$$

2 Operations on Vectors

When we are talking about vectors, a real number is called a **scalar**. There are two major vector operations:

- Vector Addition: $\mathbf{u} + \mathbf{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
- Multiplication by a scalar: $k\mathbf{u} = k \langle u_1, u_2, u_3 \rangle = \langle ku_1, ku_2, ku_3 \rangle$

In both cases, we simply apply the appropriate operation to each component seperately. Note that $\|k\mathbf{u}\| = |k| \|\mathbf{u}\|$.

We can also add vectors geometrically by drawing the first vector, drawing the second vector from the terminal point of the first one, and then drawing the resulting vector (the sum) from the initial point of the first vector to the terminal point of the second vector.

We can define vector subtraction by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$$

Example 3 (From Textbook)**.** *Let* $\boldsymbol{u} = \langle -1, 3, 1 \rangle$ *and* $\boldsymbol{v} = \langle 4, 7, 0 \rangle$ *. Find* $2\boldsymbol{u} + 3\boldsymbol{v}$*,* $\boldsymbol{u} - \boldsymbol{v}$ *and* $\left\| \frac{1}{2}\boldsymbol{u} \right\|$*.*

It is easy to show that vectors operations have the following properties: let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + 0 = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = 0$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

This shows that the set of all vectors of a certain dimension is a **vector space** or **linear space**.

3 Unit Vectors

A unit vector is a vector with magnitude 1. Three important unit vectors are the **standard unit vectors**:

$$\mathbf{i} = \langle 1,0,0\rangle \quad \mathbf{j} = \langle 0,1,0\rangle \quad \mathbf{k} = \langle 0,0,1\rangle$$

We can write any vector as a linear combination of the standard unit vectors:

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2, v_3\rangle \\ &= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle \\ &= v_1 \langle 1,0,0\rangle + v_2 \langle 0,1,0\rangle + v_3 \langle 0,0,1\rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\end{aligned}$$

For any nonzero vector $\mathbf{v}$, we can obtain a unit vector pointed in the same direction by

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

We call this the **direction** of $\mathbf{v}$.

Example 4. *Find the direction of the vector* $\langle 1,2,2\rangle$

Note that we can also express this vector as $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Its norm is $\sqrt{1+4+4} = 2$, so the direction would be given by $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$ or $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

We can always express a vector as a magnitude times a direction:

$$\mathbf{v} = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|}\mathbf{v} = \|\mathbf{v}\|\left(\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right)$$

4 The Midpoint of a line segment

If we have two points (x_1, y_1, z_1) and (x_2, y_2, z_2), the midpoint of the line segment between them is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

This can be shown using vectors.