Determinants

Definition Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the *determinant* of A is the scalar

$$\begin{aligned} \det A &= |A| \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

For any $n \times n$ matrix A, the determinant $\det A_{ij}$ of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column is called the (i, j)-*minor* of A.

Determinants of $n \times n$ Matrices

Definition Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the *determinant* of A is the scalar

$$\begin{aligned} \det A &= |A| \\ &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

It is convenient to combine a minor with its plus or minus sign. To this end, we define the *(i, j)-cofactor of* A to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

With this notation, the definition becomes

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

One of the exercises in the textbook asks you to check that this definition correctly gives the formula for the determinant of a 2×2 matrix when $n = 2$.

This definition is often referred to as *cofactor expansion along the first row*.

Theorem 4.1. The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\begin{aligned} \det A &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{1n} \\ &= \sum_{j=1}^{n} a_{ij}C_{ij} \end{aligned}$$

(which is the *cofactor expansion along the ith row*) and also as

$$\begin{aligned} \det A &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ &= \sum_{i=1}^{n} a_{ij}C_{ij} \end{aligned}$$

(the *cofactor expansion along the jth column*).

Since

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

each cofactor is plus or minus the corresponding minor, with the correct sign given by the term $(-1)^{i+j}$. A quick way to determine whether the sign is $+$ or $-$ is to remember that the signs form a “checkerboard" pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem 4.2. The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then

$$\det A = a_{11}a_{22} \cdots a_{nn}$$

Properties of Determinants

The most efficient way to compute determinants is to use row reduction. However, not every elementary row operation leaves the determinant of a matrix unchanged. The next theorem summarizes the main properties you need to understand in order to use row reduction effectively.

Theorem 4.3. Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.

b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.

c. If A has two identical rows (columns), then $\det A = 0$.

d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.

e. If A, B and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.

f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Corollary Let A, B be two square matrices of the same size.

(a) If $A \stackrel{R_i \leftrightarrow R_j}{\longrightarrow} B$, then $\det B = -\det A$.

(b) If $A \stackrel{kR_i}{\longrightarrow} B$, then $\det B = k \det A$.

(c) If $A \stackrel{R_i + kR_j}{\longrightarrow} B$ for $i \neq j$, then $\det B = \det A$.

Method for computing determinant by e.r.o.s. Reduce A to r.e.f. by e.r.o.s keeping track of the changes in determinant according to Corollary; the determinant of r.e.f., which is a triangular matrix, is the product of its diagonal elements. (We do not need to keep account of the e.r.o.s themselves; the change of the determinant can be recorded at each step, as additional coefficients arising in the same line of equalities.)

Example. We use vertical bars notation for the determinant for a change.

$$\begin{vmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix} \stackrel{R_1 \leftrightarrow R_2}{=} - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 3 & 2 \end{vmatrix} \stackrel{(1/2)R_2}{=} -2 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 3 & 2 \end{vmatrix} \stackrel{R_3 - 3R_2}{=}$$

$$-2 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & -5/2 \end{vmatrix} = -2 \cdot 1 \cdot 1 \cdot (-5/2) = 5.$$

Note that when we multiplied a row by $1/2$, the factor 2 appeared in the r.h.s., because the next determinant is $1/2$ of the previous, so the previous is 2 times the next.

Determinants of Elementary Matrices

Recall that the elementary matrix results from performing an elementary row operation on an identity matrix. Setting $A = I_n$ in Theorem 4.3 yields the following theorem.

Theorem 4.4. Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.

b. If E results from multiplying one row of I_n by k, then $\det E = k$.

c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Lemma 4.5. Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

We can use Lemma 4.5 to prove a very important theorem: the characterization of invertibility in terms of determinants.

Theorem 4.6. A square matrix A is invertible if and only if $\det A \neq 0$.

Determinants and Matrix Operations

Theorem 4.7. If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

Theorem 4.8. If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

Theorem 4.9. If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

Theorem 4.10. For any square matrix A,

$$\det A = \det(A^T)$$

Example. Suppose that A, B, C are 3×3 matrices with determinants $|A| = 5,\ |B| = 3,\ |C| = 2$. Then, e.g., $\det(2A) = 2^3 \cdot 5 = 40$;

$$\det(A^3B^2C^TA^{-1}) = \det(A)^3 \cdot \det(B)^2 \cdot \det C \cdot (1/\det A) = 5^3 3^2 2(1/5) = 5^2 3^2 2 = 450.$$

Cramer's Rule and the Adjoint

Cramer's Rule gives a formula for describing the solution of certain systems of n linear equations in n variables entirely in terms of determinants.

We will need some new notation for this result and its proof. For an $n \times n$ matrix A and a vector $\vec{b}$ in $\mathbb{R}^n$, let $A_i(\vec{b})$ denote the matrix obtained by replacing the ith column of A by $\vec{b}$. That is,

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n]$$

Theorem 4.11. Cramer's Rule

Let A be an invertible $n \times n$ matrix and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\vec{x}$ of the system $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A} \text{ for } i = 1, \ldots, n$$

Theorem 4.12. Let A be an invertible $n \times n$ matrix.

Then

$$A^{-1} = \frac{1}{\det A}[C_{ij}]^T$$

Notation:

$$[C_{ij}]^T = \operatorname{adj} A$$

(**adjoint** of A)..