Triple Integrals

DEFINITION Volume

The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. *Constant Multiple:* $\iiint_D kF\, dV = k \iiint_D F\, dV$ (any number k)
2. *Sum and Difference:* $\iiint_D (F \pm G)\, dV = \iiint_D F\, dV \pm \iiint_D G\, dV$
3. *Domination:*
   (a) $\iiint_D F\, dV \geq 0$ if $F \geq 0$ on D
   (b) $\iiint_D F\, dV \geq \iiint_D G\, dV$ if $F \geq G$ on D
4. *Additivity:* $\iiint_D F\, dV = \iiint_{D_1} E\, dV + \iiint_{D_2} F\, dV$
   if D is the union of two nonoverlapping regions D_1 and D_2.

Example 1: find the volume of the region enclosed in the lower by $z=x^2+3y^2$ and upper by $z=8-x^2-y^2$?

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

To find limits of integration : let z=0 and two equations of z as :

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

$$2x^2 + 4y^2 = 8 \rightarrow \qquad x^2 + 2y^2 = 4$$

$$y^2 = \frac{(4 - x^2)}{2} \rightarrow \qquad y = \pm\sqrt{\frac{(4 - x^2)}{2}}$$

$$x = \pm 2$$

$$V = \iiint_D dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}.$$ After integration with the substitution $x = 2\sin u$.

Mass, Moments in Three Dimensions:

Mass: $M = \iiint_D \delta \, dV \qquad (\delta = \delta(x, y, z) = \text{density})$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \, \delta \, dV, \qquad M_{xz} = \iiint_D y \, \delta \, dV, \qquad M_{xy} = \iiint_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \iiint (y^2 + z^2) \, \delta \, dV$$

$$I_y = \iiint (x^2 + z^2) \, \delta \, dV$$

$$I_z = \iiint (x^2 + y^2) \, \delta \, dV$$

Moments of inertia about a line *L*:

$I_L = \iiint r^2 \, \delta \, dV \qquad (r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L)$

Radius of gyration about a line *L*:

$$R_L = \sqrt{I_L / M}$$

EXAMPLE 1 Finding the Center of Mass of a Solid in Space

Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.34).

Solution By symmetry $\overline{x} = \overline{y} = 0$. To find $\overline{z}$, we first calculate

$$M_{xy} = \iiint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2}\iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2\,r\,dr\,d\theta \qquad \text{Polar coordinates}$$

$$= \frac{\delta}{2}\int_0^{2\pi}\left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3}\int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives

$$M = \iint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\overline{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 4/3)$. ■

When the density of a solid object is constant (as in Example 1), the center of mass is called the **centroid** of the object (as was the case for two-dimensional shapes in Section 15.2).

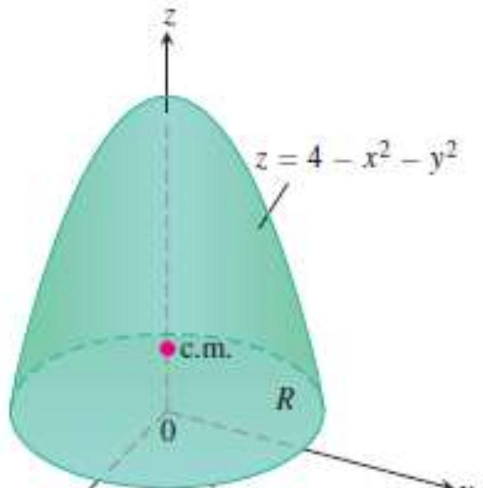

FIGURE 15.34 Finding the center of mass of a solid (Example 1).

EXAMPLE 2 Finding the Moments of Inertia About the Coordinate Axes

Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 15.35.

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$I_x = 8\int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz = 4a\delta \int_0^{c/2} \int_0^{b/2} (y^2 + z^2)\, dy\, dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

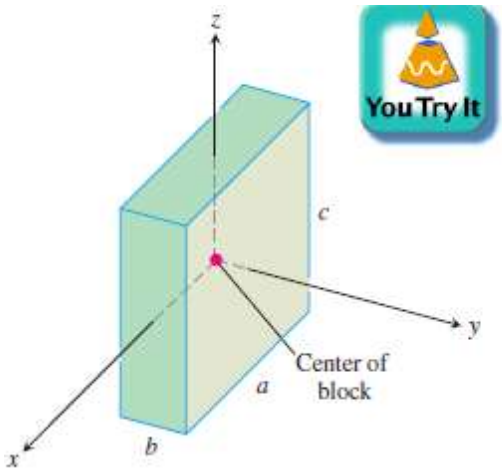

FIGURE 15.35 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 2).