## 1)<u>Uniform or Rectangular Distribution</u>

<u>Uniform distribution</u> If the probability density function of a random variable $X$ is given by

$$f_X(x) = f_X(x; a, b) = \frac{1}{b-a} I_{[a,b]}(x),$$

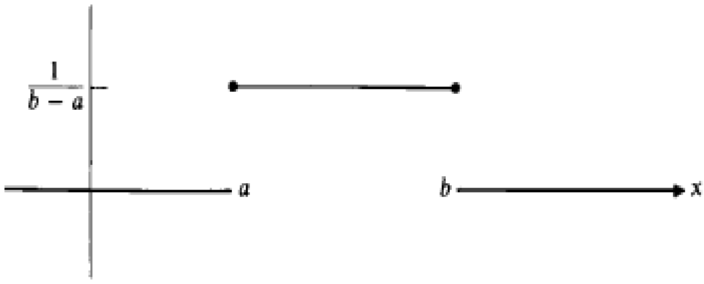


**FIGURE 1 Uniform probability density.**

where the parameters a and b satisfy $-\infty < a < b < \infty$, then the random variable $X$ is defined to be uniformly distributed over the interval $[a, b]$,

**Theorem 12** If $X$ is uniformly distributed over $[a, b]$, then

$$\mathscr{E}[X] = \frac{a+b}{2}, \quad \text{var}[X] = \frac{(b-a)^2}{12}, \quad \text{and} \quad m_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}.$$

**PROOF**

$$\mathscr{E}[X] = \int_a^b x \frac{1}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$\text{var}[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

$$m_X(t) = \mathscr{E}[e^{tX}] = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{e^{bt} - e^{at}}{(b-a)t}.$$

The uniform distribution gets its name from the fact that its density is uniform, or constant, over the interval $[a, b]$. It is also called the rectangular distribution-the shape of the density is rectangular.

The cumulative distribution function of a uniform random variable is given by

$$F_X(x) = \left(\frac{x-a}{b-a}\right) I_{[a,b]}(x) + I_{(b,\infty)}(x).$$

**Example**: If a wheel is spun and then allowed to come to rest, the point on the circumference of the wheel that is located opposite a certain fixed marker could be considered the value of a random variable $X$ that is uniformly distributed over the circumference of the wheel. One could then compute the probability that $X$ will fall in any given arc.

***2 Normal Distribution***

**Normal distribution** A random variable $X$ is defined to be normally distributed if its density is given by

$$f_X(x) = f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameters $\mu$, and $\sigma^2$ satisfy $-\infty < x < \infty$ and $\sigma^2 > 0$. That is, and $(\mu, \sigma^2) \in \{(\mu, \sigma^2): -\infty < x < \infty \text{ and } \sigma^2 > 0\}$

**Theorem 13** If $X$ is a normal random variable,

$$\mathscr{E}[X] = \mu, \qquad \text{var}[X] = \sigma^2, \qquad \text{and} \qquad m_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

**Theorem 14** If $X \sim N(\mu, \sigma^2)$, then

$$P[a < X < b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

**Remark** $\Phi(x) = 1 - \Phi(-x)$

EXAMPLE 13 Suppose that an instructor assumes that a student's final score is the value of a normally distributed random variable. If the instructor decides to award a grade of $A$ to those students whose score exceeds $\mu + \sigma$, a $B$ to those students whose score falls between $\mu$ and $\mu + \sigma$, a $C$ if a score falls between $\mu - \sigma$ and $\mu$, a $D$ if a score falls between $\mu - 2\sigma$ and $\mu - \sigma$, and an $F$ if the score falls below $\mu - 2\sigma$, then the proportions of each grade given can be calculated. For example, since

$$P[X > \mu + \sigma] = 1 - P[X < \mu + \sigma] = 1 - \Phi\left(\frac{\mu + \sigma - \mu}{\sigma}\right)$$
$$= 1 - \Phi(1) \approx .1587,$$

one would expect 15.87 percent of the students to receive $A$'s. ////

EXAMPLE 14 Suppose that the diameters of shafts manufactured by a certain machine are normal random variables with mean 10 centimeters and standard deviation .1 centimeter. If for a given application the shaft must meet the requirement that its diameter fall between 9.9 and 10.2 centimeters, what proportion of the shafts made by this machine will meet the requirement?

$$P[9.9 < X < 10.2] = \Phi\left(\frac{10.2 - 10}{.1}\right) - \Phi\left(\frac{9.9 - 10}{.1}\right)$$
$$= \Phi(2) - \Phi(-1) \approx .9772 - .1587 = .8185.$$ ////