Solution Let a r.v. X is c.r.v. then

1) $E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx = rth$ central moment about the origin, $r = 1,2,3, \ldots$ Therefor $E[X] = \mu$ = mean of the r.v,.

EXAMPLE 15 Let X be a random variable with probability density given by $f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$; then

$$\begin{aligned} \text{Var}[X] &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx \\ &= \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\,dx \\ &= \frac{1}{\lambda^2}. \end{aligned}$$ ////

if X is continuous with probability density function $f_X(x)$.

(iii)
$$\mathscr{E}[X] = \int_0^{\infty} [1 - F_X(x)]\,dx - \int_{-\infty}^{0} F_X(x)\,dx \qquad (6)$$

for an arbitrary random variable X. ////

EXAMPLE 11 Let X be a continuous random variable with probability density function $f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$.

$$\mathscr{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}.$$

The corresponding cumulative distribution function is

$$F_X(x) = (1 - e^{-\lambda x}) I_{[0,\infty)}(x); \text{ so } \mathscr{E}[X] = \int_0^{\infty} [1 - F_X(x)]\,dx$$

$$- \int_{-\infty}^{0} F_X(x)\,dx = \int_0^{\infty} (1 - 1 + e^{-\lambda x})\,dx = 1/\lambda.$$ ////

The variance of the r.v. X be can found directly as

$$\begin{aligned} \mathrm{E}(X-\mu)^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\ &= \int_{-\infty}^{\infty} (x^2 - \mu^2) f(x)dx \\ &= \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} dx - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \end{aligned}$$

Where $\int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$

EXAMPLE 12 Let X be a random variable with cumulative distribution function given by $F_X(x) = (1 - pe^{-\lambda x})I_{[0,\infty)}(x)$; then

$$\mathscr{E}[X] = \int_0^\infty [1 - F_X(x)]\,dx - \int_{-\infty}^0 F_X(x)\,dx = \int_0^\infty pe^{-\lambda x}\,dx = \frac{p}{\lambda}.$$

Here, we have used Eq. (6) to find the mean of a random variable that is partly discrete and partly continuous. ////

ملاحظه

غند تطبيق هذا القانون الخاص بالمتغير المستمر علينا مراعاة مجال المتغير فأذا لم يحتوي مجاله على قيم سالبه نأخذالمجال الموجود في المسأله اي الموجب فقط)

2) $E(X-\mu)^r =$
$\int_{-\infty}^{\infty}(x-\mu)^r f(x)dx = rth$ central moment about the mean

If $= 2$, then $E(X-\mu)^2 = var(X) = \sigma^2$.

3) $E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx = M_X(t) = M(t)$ moment generating function

We know that $e^{tx} = \sum_{j=0}^{\infty}\frac{(tx)^j}{j!}$, then

$$E[e^{tX}] = E\left[\sum_{j=0}^{\infty}\frac{(tX)^j}{j!}\right] = 1 + \frac{tE[X]}{1!} + \frac{tE[X^2]}{2!} \dots$$

That means this function contains the moments $E[X^r], r = 1,2,3, \dots$ which we can generate as follows:

$$\left.\frac{d^{(r)}M(t)}{dt^r}\right|_{t=0} = E[X]^r, r \in \mathbb{Z}$$

$$\left.\frac{dM(t)}{dt}\right|_{t=0} = E[X] \;, \left.\frac{d^{(2)}M(t)}{dt^2}\right|_{t=0} = E[X]^2, \dots$$

4) $E\left[e^{itX}\right] = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \phi_X(t)$ is called the characteristic function.

5) $E[t^X] = \int_{-\infty}^{\infty} t^x f(x)dx$ is called the Factorial moment generating function.

EXAMPLE 19 Suppose X has a discrete density function given by

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x = 0, 1, 2, \ldots.$$

Then

$$\mathscr{E}[t^X] = \sum_{x=0}^{\infty} \frac{t^x e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}e^{\lambda t} = e^{\lambda(t-1)}.$$

$$\frac{d}{dt}\mathscr{E}[t^X] = \frac{d}{dt}e^{\lambda(t-1)} = \lambda e^{\lambda(t-1)}; \qquad \text{hence} \quad \frac{d}{dt}\mathscr{E}[t^X]\Big|_{t=1} = \lambda.$$

6) $E[X!] = \int_{-\infty}^{\infty} x!\, f(x)dx$ is called the Factorial moment that can be used in finding the raw moment for Poisson, Binomial, Negative Binomial,.. distributions.

Now let a r.v. X is c.r.v. then

$$E[X^r] = \sum_{\forall x} x^r P(x)$$

$$E[(X-\mu)^r] = \sum_{\forall x}(x-\mu)^x P(x)$$

EXAMPLE 17 Let X be a random variable with probability density function given by $f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$.

$$m_X(t) = \mathscr{E}[e^{tX}] = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - t} \qquad \text{for } t < \lambda.$$

$$m'(t) = \frac{dm(t)}{dt} = \frac{\lambda}{(\lambda - t)^2} \qquad \text{hence} \quad m'(0) = \mathscr{E}[X] = \frac{1}{\lambda}.$$

And $$m''(t) = \frac{2\lambda}{(\lambda - t)^3}, \qquad \text{so } m''(0) = \mathscr{E}[X^2] = \frac{2}{\lambda^2}. \qquad ////$$

$$E[t^x] = \sum_{\forall x} t^x P(x)$$

EXAMPLE 10 Consider the experiment of tossing two dice. Let X denote the total of the two dice and Y their absolute difference. The discrete density functions for X and Y are given in Example 6.

$$\mathscr{E}[Y] = \sum y_j f_Y(y_j) = \sum_{i=0}^{5} i f_Y(i) = 0 \cdot \tfrac{6}{36} + 1 \cdot \tfrac{10}{36} + 2 \cdot \tfrac{8}{36} + 3 \cdot \tfrac{6}{36} + 4 \cdot \tfrac{4}{36} + 5 \cdot \tfrac{2}{36} = \tfrac{70}{36}.$$

$$\mathscr{E}[X] = \sum_{i=2}^{12} i f_X(i) = 7.$$

Note that $\mathscr{E}[Y]$ is not one of the possible values of Y. ////

Where

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Therefore

y	0	1	2	3	4	5
P(y)	$\frac{6}{36}$	$\frac{8}{36}$	$\frac{8}{36}$	$\frac{2}{36}$	$\frac{5}{36}$	$\frac{6}{36}$

Introduction to the Theory Of Statistics

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