Some Distributions

Discrete Distributions

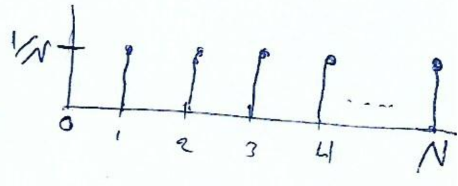
① uniform dis

[a, a+1, ..., b]

A random variable X haveing a density funiction

$$f(x) = f(x; N) = \frac{1}{N} \quad \text{for } x = 1, 2, \ldots, N$$
$$0 \quad \text{otherwise}$$

is called a discrete uniform random variable.



Theorem If X has a discrete uniform dis, then $E(X) = (N+1)/2$

$$var(x) = \frac{(N^2-1)}{12}$$

and $m_X(t) = E(e^{tX}) = \sum_{j=1}^{N} e^{jt} \frac{1}{N}$.

proof

$$E(X) = \sum_{j=1}^{N} j \frac{1}{N} = \frac{N+1}{2}.$$

$$var(x) = E(x^2) - [E(x)]^2 = \sum_{j=1}^{N} j^2 \frac{1}{N} - \left(\frac{N+1}{2}\right)^2$$

$$= \frac{N(N+1)(2N+1)}{6N} - \frac{(N+1)^2}{4} = \frac{(N+1)(N-2)}{12}$$

$$E(e^{tX}) = \sum_{j=1}^{N} e^{jt} \frac{1}{N}, \quad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a+1}{n} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Example Throwing a dice

$$E(x) = \frac{1+6}{2} = 3.5 \quad \& \quad var(x) = \frac{6^2-1}{12} \approx 2.92$$

② Bernoulli and Binomial dis.

def Bernoulli dis. A random variable X is defined to have a Bernoulli dis. if the discrete density fun of X is given by

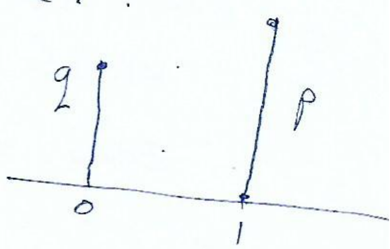
$$f_X(x) = f_X(x; p) = \begin{cases} p^x(1-p)^{1-x} & \text{for } x=0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= p^x(1-p)^{1-x} I_{(0,1)}(x),$$

where the parameter p satisfies $0 \le p \le 1$.

$1-p$ is often denoted by q.

$(X \sim Ber(p))$



Theorem If X has a Bernoulli dis, then

$E(X) = p$, $var(X) = pq$ and $m_X(t) = pe^t + q$.

proof

$E(X) = 0 \cdot q + 1 \cdot p = p$

$var(X) = E(X^2) - [E(X)]^2 = 0^2 \cdot q + 1^2 \cdot p - p^2 = pq$.

$m_X(t) = E[e^{tX}] = q + pe^t$.

Remark ① A random experiment whose outcomes have been classi[fied] into two categories, called "success" and "failure" ~~respectively~~ represented [by] the letter P and q, respectively, is called a Bernoulli trial.

② If a random variable X is defined as 1 if a Bernoulli trial result[s] in success and 0 if the same Bernoulli trial results in failure. Then X has a Bernoulli dis with parameter $p = P[\text{success}]$.

Example $X \sim Ber(0.95)$ find $p(x)$, μ_X, $Var(X)$ $M_X(t)$

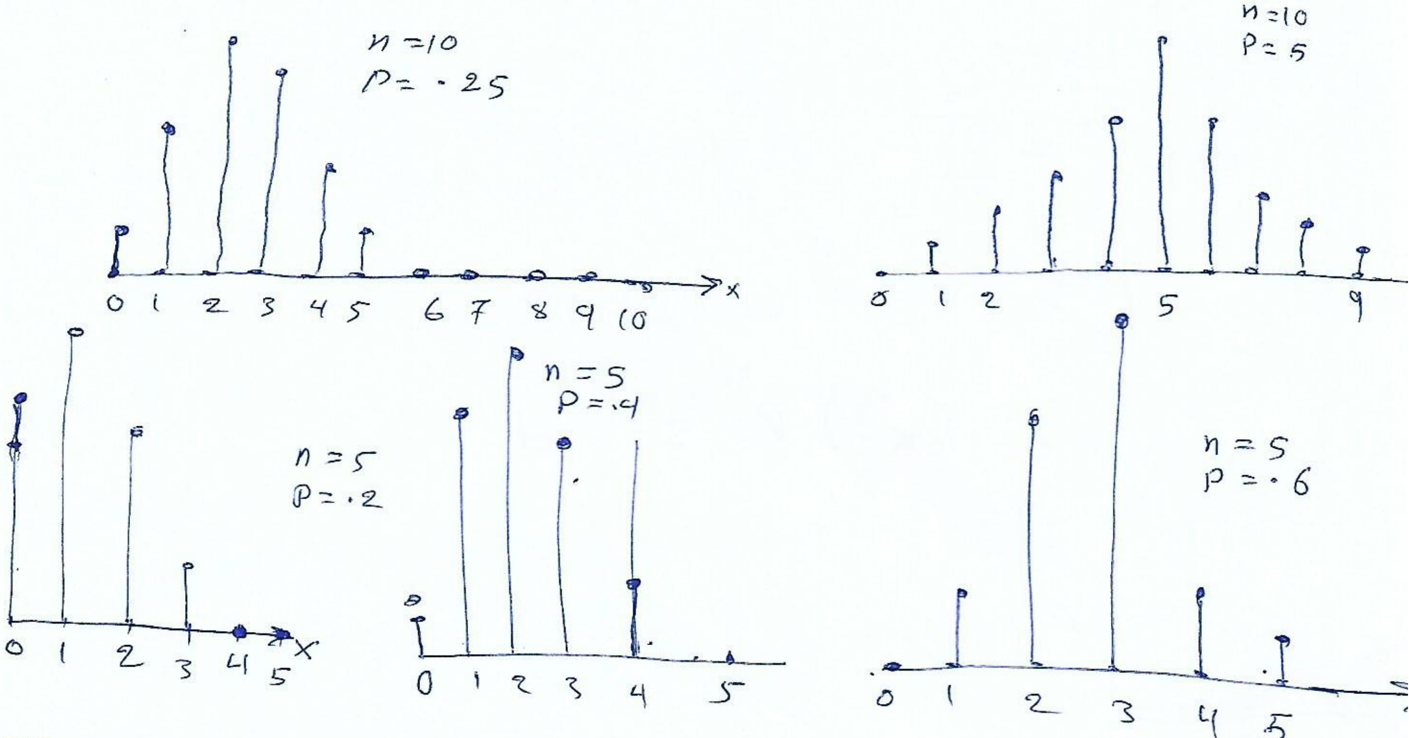
Example A result of a toss of a coin with head, say, equal to one and Ta[il] equal to zero

Note p is called the prob of "success" and $q = 1-p$ the prob of "failur[e]"

Def Binomial dis . A random variable X is defined to have a binomial if the discrete ~~density~~ mass fun of X is given by

$$f_X(x) = f_X(x;n,p) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

$$= \binom{n}{x} p^x q^{n-x}$$



Theorem If X has a binomial dis, then $E(X) = np$, $var(X) = npq$ and $m_X(t) = (q + pe^t)^n$.

proof

$$m_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n.$$

Now, $m'_X(t) = npe^t (pe^t + q)^{n-1}$

and $m''_X(t) = n(n-1)(pe^t)^2 (pe^t + q)^{n-2} + npe^t (pe^t + q)^{n-1}$,

hence $E(X) = m'_X(0) = np$. & $var(X) = E(X^2) - [E(X)]^2$

$\therefore var = m''_X(0) - (np)^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$.

Remark The Binomial dis reduces to the Bernoulli dis when $n = 1$. Sometime the Bernolli dis is called the point binomial.

ملاحظة ① اذا كانت $p<q$ يعني ان التوزيع ذو التواء موجب

" " $p>q$ " " " " " سالب

" " " " $p=q$ " " " " متماثل

نظرية ثنائي الحدين Binomial theorem تنص على ما يلي

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k}$$

Let $a=p$, $b=q$

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$p+q=1 \rightarrow (p+q)^n = 1$, $\because$ n is know

Then $\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = 1$

(دالة التوزيع غير سالبة ووحيدة القيمة)

$$F(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k}$$

غير مطلوب الاثبات

Theorem

$\mu_x = np$, $var(x) = npq$

$$\mu = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!\, p^{x-1} q^{n-x}}{(x-1)!\,(n-x)!}$$

$y = x-1$, $n' = n-1$

$$\mu = np \sum_{x=0}^{n'} \binom{n'}{y} p^y q^{n-y}$$

$= np$, $y \sim b(n', p)$

So , $Ex^2 - (Ex)^2 = Var($

$$Ex^2 = \sum_{x=0}^{n} x^n \binom{n}{x} p^x q^{n-x}$$

$x^2 = x(x-1) + 1$

$$Ex^2 = \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x q^{n-x}$$

$+np$

$$n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x} + np$$

$n^* = n-2$, $y = x-2$

$$Ex^2 = n(n-1)p^2 \sum_{y=0}^{n^*} \binom{n^*}{y} p^y q^{n^*-y} + np$$

$= n(n-1)p^2 + np$, $y \sim b(n^*, p)$

$\sigma_x^2 = n(n-1)p^2 + np - n^2p^2 = npq$

Then $\sigma = \mu_x q$

$var < \mu$