Random Experiment

It is an experiment that its outcomes can not be predicted, and each its outcome has the same chance(probability).

Definition 11 Sample space The *sample space*, denoted by Ω, is the collection or totality of all possible outcomes of a conceptual experiment.

Definition 12 Event and event space An *event* is a subset of the sample space. The class of all events associated with a given experiment is defined to be the *event space*. ////

Example

Tossing one single fair coin one time.

Sample space=$\boldsymbol{\Omega} = \{H, T\}$

Let $\boldsymbol{X}$ be number of appearance the head, that $P(H) = \frac{1}{2}$ then $P(T) = \frac{1}{2}$

Now let $\boldsymbol{X = 1}$ for appearance the head**,** and let $\boldsymbol{X = 0}$ for without appearance the head**,** e.g. $P(H) = P(X = 1) = \frac{1}{2}$ then

$$P(T) = P(X = 0) = \frac{1}{2} \text{ and } P(X = 1) + P(X = 0) = \mathbf{1}$$

Thus $\boldsymbol{X}$ is discrete random variable since its values are countale values, so that its probability is called probability mass function, th\at is

$$P(X = x) = \begin{cases} \frac{1}{2}, & x = 0,1 \\ 0 & other\ wise \end{cases}$$

Or $P(X = x) = \frac{1}{2} I_{\{0,1\}}(x)$ where

Definition 14 Indicator function Let Ω be any space with points ω and A any subset of Ω. The *indicator function* of A, denoted by $I_A(\cdot)$, is the function with domain Ω and counterdomain equal to the set consisting of the two real numbers 0 and 1 defined by

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

$I_A(\cdot)$ clearly "indicates" the set A. ////

EXAMPLE 13 Let the function $f(\cdot)$ be defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ x & \text{for} \quad 0 < x \le 1 \\ 2 - x & \text{for} \quad 1 < x \le 2 \\ 0 & \text{for} \quad 2 < x. \end{cases}$$

By using the indicator function, $f(x)$ can be written as

$$f(x) = xI_{(0,1]}(x) + (2 - x)I_{(1,2]}(x).$$

***Boolen Algebra* (*Algebra*)**

It is a set of all possible subsets of the sample space Ω,denoted as $\boldsymbol{\mathcal{A}}$, which are called events. Therefore this set has the following properties:

1) $\Omega \in \boldsymbol{\mathcal{A}}$,

2) $\boldsymbol{A} \in \boldsymbol{\mathcal{A}} \to \boldsymbol{A}^c \in \boldsymbol{\mathcal{A}}$,

3) $\boldsymbol{A_1}, \boldsymbol{A_2} \in \boldsymbol{\mathcal{A}} \to \boldsymbol{A_1} \cup \boldsymbol{A_2} \in \boldsymbol{\mathcal{A}}$.

***Sigma-algebra*($\boldsymbol{\sigma}$ −algebra)**

Let $\boldsymbol{A}$ be non empty set. Then a collection of all subsets of $\boldsymbol{A}$, denoted $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{F}}$,..

It is called $\boldsymbol{\sigma}$ **−algebra** iff it has the following properties:

1) $\Omega \in \boldsymbol{\mathcal{A}}$,

2) $\boldsymbol{A} \in \boldsymbol{\mathcal{A}} \to \boldsymbol{A}^c \in \boldsymbol{\mathcal{A}}$,

3) $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{A_3}, \ldots. \in \boldsymbol{\mathcal{A}} \to \boldsymbol{A_1} \cup \boldsymbol{A_2} \cup \boldsymbol{A_3} \cup \ldots. = \bigcup_{i=1}^{\infty} \boldsymbol{A_i} \in \boldsymbol{\mathcal{A}}$.

EXAMPLE 7 Toss a penny, nickel, and dime simultaneously, and note which side is up on each. There are eight possible outcomes of this experiment. Ω ={(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)}. We are using the first position of (·, ·, ·), called a 3-*tuple*, to record the outcome of the penny, the second position to record the outcome of the nickel, and the third position to record the outcome of the dime. Let A_i = {exactly i heads}; i = 0, 1, 2, 3. For each i, A_i is an event. Note that A_0 and A_3 are each elementary events. Again all subsets of Ω are events; there are $2^8 = 256$ of them. ////

Definition 15 Probability function A *probability function* $P[\cdot]$ is a set function with domain $\mathscr{A}$ (an algebra of events)* and counterdomain the interval [0, 1] which satisfies the following axioms:

(i) $P[A] \geq 0$ for every $A \in \mathscr{A}$.

(ii) $P[\Omega] = 1$.

(iii) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events in $\mathscr{A}$

(that is, $A_i \cap A_j = \phi$ for $i \neq j$; $i, j = 1, 2, \ldots$) and if $A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$, then $P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P[A_i]$. ////

where

Definition 10 Disjoint or mutually exclusive Subsets A and B of Ω are defined to be *mutually exclusive* or *disjoint* if $A \cap B = \phi$. Subsets $A_1, A_2, \ldots$ are defined to be *mutually exclusive* if $A_i A_j = \phi$ for every $i \neq j$.

EXAMPLE 16 Consider the experiment of tossing two coins, say a penny and a nickel. Let $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$ where the first component of $(\cdot, \cdot)$ represents the outcome for the penny. Let us model this random experiment by assuming that the four points in Ω are equally likely; that is, assume $P[\{(H, H)\}] = P[\{(H, T)\}] = P[\{(T, H)\}] = P[\{(T, T)\}]$. The following question arises: Is the $P[\cdot]$ function that is implicitly defined by the above really a probability function; that is, does it satisfy the three axioms? It can be shown that it does, and so it is a probability function.

Definition 16 Probability space A *probability space* is the triplet $(\Omega, \mathscr{A}, P[\cdot])$, where Ω is a sample space, $\mathscr{A}$ is a collection (assumed to be an algebra) of events (each a subset of Ω), and $P[\cdot]$ is a probability function with domain $\mathscr{A}$. ////

"Introduction to the Theory of Statistics"

By

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