Kinematics in Three Dimension (3-D)

4.1 Position and Displacement

The purpose of this section is to generalize the previously introduced concept of displacement, velocity and acceleration in order to deal with motion in three dimension.

A position vector $\vec{r}$ can be written as:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad (1)$$

where $x\hat{\imath}, y\hat{\jmath}$ and $z\hat{k}$ are the vector components of $\vec{r}$. A particle moves from **A** with position vector $\vec{r}_i$ to **B** the position vector $\vec{r}_f$ during a certain time interval, as shown in figure 1, then the particle's displacement $\Delta\vec{r}$ is :

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i \qquad (2)$$

$$\Delta\vec{r} = \left(x_f\hat{\imath} + y_f\hat{\jmath} + z_f\hat{k}\right) - \left(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}\right)$$

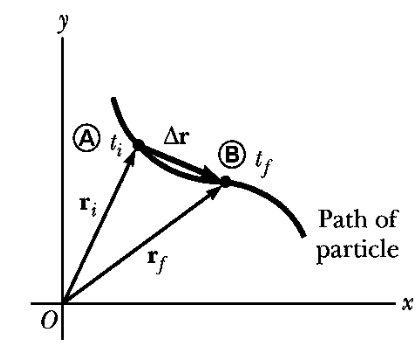
$$\Delta\vec{r} = \left(x_f - x_i\right)\hat{\imath} + \left(y_f - y_i\right)\hat{\jmath} + \left(z_f - z_i\right)\hat{k} \qquad (3)$$

Or

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k} \qquad (4)$$

where

$\Delta x = x_f - x_i$, $\Delta y = y_f - y_i$ and $\Delta z = z_f - z_i$



4.2 Velocity and Acceleration

Consider a particle in a plane moving from P to Q the change in position vector is $\Delta\vec{r}$:

$$\Delta\vec{r} = \Delta x\hat{\imath} + \nabla y\hat{\jmath} \qquad (5)$$

Let Δt be the time interval for the motion from P to Q

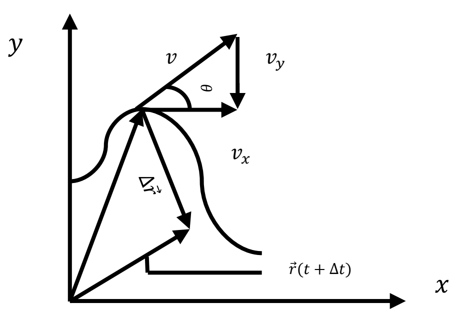
The average velocity of the particle is then defined as the vector quantity equal to the displacement divided by the time interval:

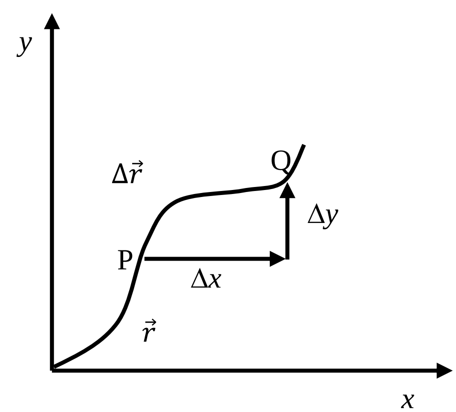
$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k} \qquad (6)$$

Let $\bar{v}_x = \frac{\Delta x}{\Delta t}$, $\bar{v}_y = \frac{\Delta y}{\Delta t}$ and $\bar{v}_z = \frac{\Delta z}{\Delta t}$

$\bar{v} = \bar{v}_x + \bar{v}_y + \bar{v}_z$

Instantaneous velocity, v at the point P is defined in magnitude and direction as the limit approached by the average velocity when point Q is taken to be closer and closer to P (as $\Delta t \to 0$)





Direction of $\vec{r}$ is tangent to path of particle of P.

$\vec{v} = \frac{d}{dt}[x\hat{\imath} + y\hat{\jmath} + z\hat{k}]$

$= \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

Magnitude of $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

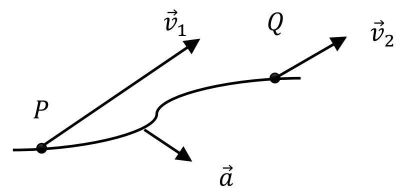
$$tan\theta = \frac{v_y}{v_x}$$

The average acceleration, $\vec{a}_{av}$, of the particle as it moves from P to Q is defined as the vector change in velocity, $\Delta\vec{v}$ divided by the time interval Δt.

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

$$= \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} + \frac{\Delta v_z}{\Delta t}\hat{k}$$

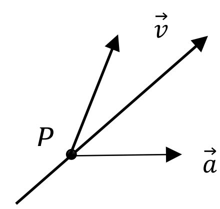
$$= a_{x\ av}\hat{\imath} + a_{y\ av}\hat{\jmath} + a_{z\ av}\hat{k}$$



The instantaneous acceleration, $\vec{a}$, at point P is defined in magnitude and direction as the limit approached by the average acceleration when point Q approaches point P and Δv and Δt both approach zero

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$



Where

$$a_x = \frac{dv_x}{dt} = \frac{d^2\vec{x}}{dt^2}$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2\vec{y}}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2\vec{z}}{dt^2}$$

4.3 Motion at Constant Acceleration

If initial velocity is $\vec{v}_0$, then the velocity after time t is

$\vec{v} = \vec{v}_0 + \vec{a}t$

The $x, y\ and\ z$ components are

$\vec{v}_x = \vec{v}_{0x} + \vec{a}_x t$, $\vec{v}_y = \vec{v}_{0y} + \vec{a}_y t$ and $\vec{v}_z = \vec{v}_{0z} + \vec{a}_z t$

Using arguments as in the one. Dimension each the position vector become

$\vec{r} = \vec{r}_0 + v_0 t + \frac{1}{2}\vec{a}t$

In components:

$x(t) = x_0 + v_{ox}t + \frac{1}{2}a_x t^2$

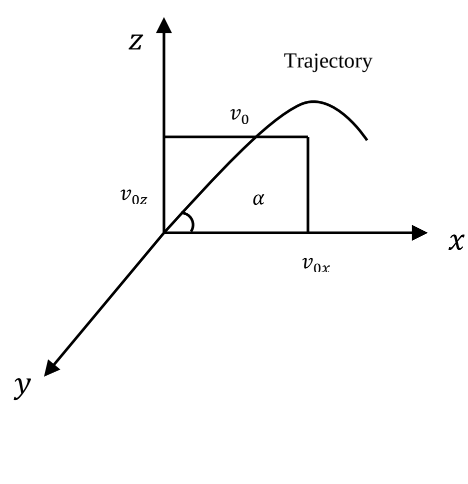
$y(t) = y_0 + v_{oy}t + \frac{1}{2}a_y t^2$

$z(t) = z_0 + v_{oz}t + \frac{1}{2}a_z t^2$

4.4 Motion of Projectiles

Study of motion of a body which is given some initial velocity and starts from some initial position and follows a path determined by the effect of the gravitational acceleration and by air resistance. Projectile path is called its trajectory.

Freely falling body near the earth's surface experiences downward acceleration with $g = 9.81\ m/s^2$



We the motion is in the *xz*-plane, *z*-axis is in the direction of the upward vertical, x-axis is in the direction of the horizontal velocity, $a_x = 0$, $a_y = 0$ and

$a_z = -g = -9.81\,\frac{m}{s^2}$ [acceleration apposite to +z]

Equation of motion:

$x(t) = x_0 + v_{0x}t$ $\qquad v_x(t) = v_{0x}$

$y(t) = 0$ $\qquad v_y(t) = 0$

$z(t) = z_0 + v_{0z}t - \frac{1}{2}gt^2$ $\qquad v_z(t) = v_{0z} - gt$

That is, motion are decoupled. Motion along each axis is independent of motion along other axis. Can treat separately chosen coordinates such that y, v_{0y} and a_y are initially zero and remain that way.

4.5 Mathematical form of ballistic trajectory

$x(t) = x_0 + v_{0x}t \Longrightarrow t = \frac{x-x_0}{v_{0x}}$

Substitute for t in $z(t)$

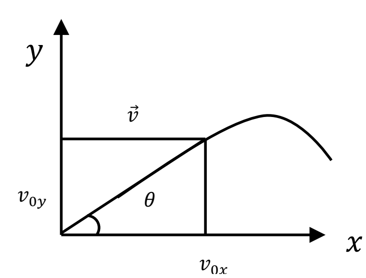
$$z = z_0 + v_{oz}\left(\frac{x-x_0}{v_{0x}}\right) - \frac{1}{2}g\left(\frac{x-x_0}{v_{0x}}\right)^2$$

$z = a + bx + cx^2$, where a, b and c are constant [equation of parabola]

Horizontal and vertical motion

$x(t) = x_0 + v_{0x}t$

$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$



But

$v_{0x} = v_0 cos\theta$

$v_{0y} = v_0 sin\theta$

Then,

$x(t) = x_0 + (v_0 cos\theta)t$

$y(t) = y_0 + (v_0 sin\theta)t - \frac{1}{2}gt^2$

$\vec{v}_x(t) = \frac{dx}{dt} = v_0 cos\theta$

$\vec{v}_y(t) = v_0 sin\theta - gt$

The Horizontal Range: to find the horizontal range R or x_{max}, let us put $x - x_0 = R$ and $y - y_0 = 0$, obtaining

$R = v_0 cos\theta t$ (1)

$0 = v_0 sin\theta t - \frac{1}{2}gt^2$ (2)

substituting Eq.2 in Eq.1 yield

$R = \frac{2v_0^2 sin\theta cos\theta}{g} = \frac{v_0^2 sin2\theta}{g}$

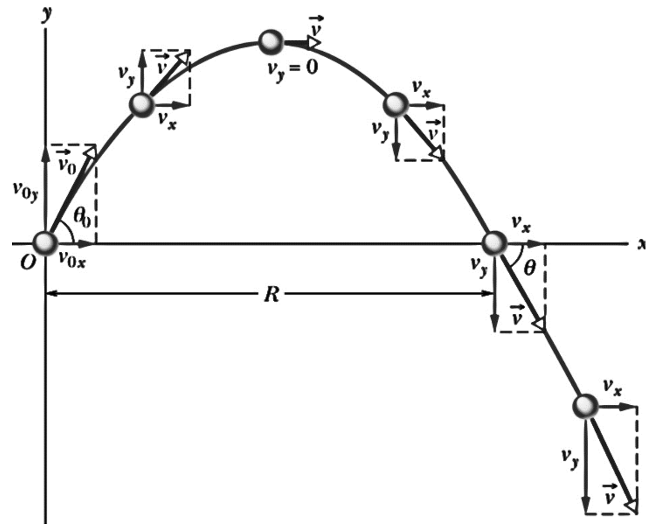
Where $sin2\theta = 2\, sin\theta\, cos\theta$

We can determined the maximum height (h), if we put $h = (y - y_0)$ and $v_y = 0$, then

$v_y^2 = 0 = (v_0 sin\theta)^2 - 2gh$

$\therefore h = \frac{v_0^2\, sin^2\theta}{2g}$ (maximum height)

$t_{flight} = \frac{2v_0 sin\theta}{g}$ (flight time) all these specific result (height, time, range) apply only if launch and impact points are at the same height, y. Special cases must be treated carefully



maximum Range occurs at angle $\theta = 45$, the maximum value of $sin2\theta = 1$

$$R = \frac{v_0^2 sin2\theta}{g} = \frac{v_0^2}{g}$$

Example (1): The coordinates of a particle moving in the *xy* - plane are given by

$$x = 1 + 2t^2 \qquad (m)$$

$$y = 2t + t^3 \qquad (m)$$

Find the particle's position, velocity and acceleration at time $t = 2s$

Solution:

$t = 25$, $x = 1 + 2(2)^2 = 9m$

$$y = 2(2) + (2)^3 = 12m$$

$\vec{r} = 9\hat{\imath} + 12\hat{\jmath}$

distance from origin

$$r = \sqrt{x^2 + y^2} = \sqrt{9^2 + 12^2} = 15m$$

$$tan\theta = \frac{x}{y} \Longrightarrow \theta = tan^{-1}\frac{y}{x} = tan^{-1}\frac{12}{9} = 53.1^{\circ}$$

The velocity

$$v_x = \frac{dx}{dt} = 4t \left(\frac{m}{s}\right) \qquad v_y = \frac{dy}{dt} = 2 + 3t^2 \left(\frac{m}{s}\right)$$

At $t = 2s$ $v_x(2) = 8\left(\frac{m}{s}\right)$ $v_y(2) = 14\left(\frac{m}{s}\right)$

$\vec{v}(t = 2) = 8\hat{\imath} + 14\hat{\jmath}$

$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{8^2 + 14^2} = 16\left(\frac{m}{s}\right)$

$\theta_v = tan^{-1}\frac{v_y}{v_x} = tan^{-1}\frac{14}{8} \Longrightarrow \theta_v = 60.3^{\circ}$

Acceleration

$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 4\left(\frac{m}{s^2}\right)$ $a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 6t\left(\frac{m}{s^2}\right)$

At $t = 2s$ $a_x = 4\left(\frac{m}{s^2}\right)$ $a_y = 12\left(\frac{m}{s^2}\right)$

$\vec{a} = 4\hat{\imath} + 12\hat{\jmath}$

$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 12^2} = 12.6\left(\frac{m}{s^2}\right)$

$\theta_a = tan^{-1}\frac{a_y}{a_x} = tan^{-1}\frac{12}{4} \Longrightarrow \theta_a = 71.6^{\circ}$

Example (2): if the position of a particle $(\vec{r})$ moving in *xy*-plane is given by

$\vec{r} = (3t^3 - 5t)\hat{\imath} + (6 - 7t^4)\hat{\jmath}$ Calculate $\vec{r}, \vec{v}$ and $\vec{a}$ at $t = 2s$?

Solution:

$\vec{r} = (3t^3 - 5t)\hat{\imath} + (6 - 7t^4)\hat{\jmath}$

$\vec{r} = (3(2)^2 - 5(2))\hat{\imath} + (6 - 7(2)^2)\hat{\jmath}$ $at\ \ t = 2$

$\vec{r} = 4\hat{\imath} - 106\hat{\jmath}$

$\vec{v} = \frac{dr}{dt} = (9t^2 - 5)\hat{\imath} + (-28t^3)\hat{\jmath}$

$\vec{v} = (9(2)^2 - 5)\hat{\imath} + (-28(2)^3)\hat{\jmath}$ $at\ \ t = 2$

$\vec{v} = 31\hat{\imath} - 224\hat{\jmath}$

$\vec{a} = \frac{d^2r}{dt^2} = \frac{dv}{dt} = (18t)\hat{\imath} + (-28 \times 3t^2)\hat{\jmath}$

$\vec{a} = (18 \times 2)\hat{\imath} + (-28 \times 3(2)^2)\hat{\jmath}$ $\quad$ $at\ \ t = 2$

$\vec{a} = 36\hat{\imath} - 336\hat{\jmath}$

Example (3): A particle velocity $\vec{v}_0 = -2\hat{\imath} + 4\hat{\jmath}$ at $t = 0$ undergoes constant $\vec{a}$ of magnitude $a = 3\left(\frac{m}{s^2}\right)$ at angle $\theta = 130$ from the positive direction of the x-axis, what is the particle's velocity $\vec{v}$ at $t = 5\ s$, in unit vector notation, and in magnitude angle notation?

Solution:

$v_x = v_{0x} + a_x t$

$v_y = v_{0y} + a_y t$

$a_x = a\ cos\theta = 3cos130 = -1.93\left(\frac{m}{s^2}\right)$

$a_y = asin\theta = 3sin130 = 2.30\left(\frac{m}{s^2}\right)$

At $t = 5\ s$

$v_x = -2 + (-1.93)(5) = -11.65\left(\frac{m}{s}\right)$

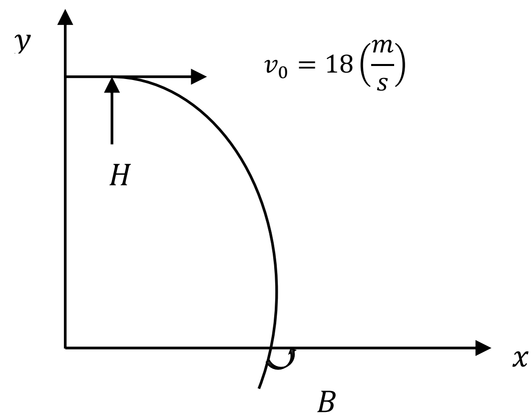
$v_y = 4 + 2.30(5) = 15.5\left(\frac{m}{s}\right)$

$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} = -11.65\hat{\imath} + 15.5\hat{\jmath}$

The magnitude of $\vec{v}$, $v = |v| = \sqrt{v_x^2 + v_y^2} = 19\left(\frac{m}{s}\right)$

$$\theta = tan^{-1}\frac{v_y}{v_x} = tan^{-1} = \frac{15.5}{-11.65} \Longrightarrow \theta = 127^{\circ}$$

Example (4): Ball kicked horizontally at $18\left(\frac{m}{s}\right)$ off a $50m$ high cliff, find



a. Time to impact
b. Speed at impact
c. Impact point
d. Angle at impact

Solution:

$$x(t) = v_0 cos\theta\ t$$

$$= v_0 t \qquad [\theta = 0^{\circ}] \qquad (1)$$

$$y(t) = y_0 + v_{0y} - \frac{1}{2}gt^2$$

$$= H - \frac{1}{2}gt^2 \qquad (2)$$

At impact we must have $y = 0$, $t = T$, solving Eq. (2)

$$T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2\times 50}{9.81}} = 3.19s$$

$$x(T) = 18 \times 3.19 = 57.42 \qquad (Eq.(1)$$

$$v_x(T) = \frac{dx}{dt} = v_0 = 18\left(\frac{m}{s}\right)$$ [independent of time]

$$v_y(T) = \frac{dy}{dt} = -gt = -9.81 \times 3.19 = -31.26\left(\frac{m}{s}\right)$$

$$tan\beta = \frac{v_y}{v_x} = \frac{-31.26}{18} \Longrightarrow \beta = -60.1$$

$$|v| = \sqrt{v_x^2 + v_y^2} = \sqrt{18^2 + (-31.26)^2} = 36.1\left(\frac{m}{s}\right)$$ speed

Example (5): Gun fired a bullet with velocity $200\left(\frac{m}{s}\right)$ by an 40° with horizontal, find a velocity and position of a bullet after $20\ s$ and find range and time required to return to ground?

Solution:

$$v_{0x} = v_0 cos\theta = 200cos40 = 153.2\left(\frac{m}{s}\right)$$

$$v_{0y} = v_0 sin\theta = 200sin40 = 128.6\left(\frac{m}{s}\right)$$

$$v_{0x} = v_x = 153.2\left(\frac{m}{s}\right)$$

$$v_y = v_{0y} - gt = 128.6 - 9.8t \qquad at\ t = 20\ s$$

$$v_y = 128.6 - 9.8(20)$$

$$v_y = -67.4\left(\frac{m}{s}\right)$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{153.2^2 + (-67.4)^2}$$

$$= 167\left(\frac{m}{s}\right)$$

$$x = 3064m\ ,\ y = 612m$$

$$t = \frac{2v_{0y}}{g} = \frac{2(128.6)}{9.8} = 26.24\ s$$

$$R = \frac{v_0^2 sin2\theta}{g} = \frac{(200)^2 sin2(40)}{9.8} = 4021m$$

$$h = \frac{v_0^2 sin^2\theta}{2g} = \frac{(200)^2(sin40)^2}{2\times 9.8} = 843.7m$$

Example (6): A long- jumper leaves the ground at an angle of 20°above the horizontal and at a speed of $11\left(\frac{m}{s}\right)$

a. How far does he jump in horizontal direction?

b. What is maximum height reached?

Solution:

a. $R = \frac{v_0^2 sin2\theta}{g} = \frac{(11)^2 sin2(20)}{9.8} = 7.936\, m$

b. $h = \frac{v_0^2 sin^2\theta}{2g} = \frac{(11)^2 sin^2 20}{2\times 9.8} = 0.722\, m$

Example (7): A Stone is thrown from the top of a building upward at an angle of 30° to the horizontal and with an initial speed of $20\left(\frac{m}{s}\right)$. If the height of the building is $45m$,

a. How long is it before the stone hits the ground

b. What the speed of the stone just before it strikes the ground?

Solution:

a. $v_{0x} = v_0 cos\theta = 20 \times cos30 = 17.3\left(\frac{m}{s}\right) = v_x$

$v_{0y} = v_0 sin\theta = 20 \times sin30 = 10\left(\frac{m}{s}\right)$

$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 \Longrightarrow -45 = 10t - \frac{1}{2}(9.8)t^2 \Longrightarrow t = 4.22s$

b. $v_x = 17.3\left(\frac{m}{s}\right)$, $v_y = v_{0y} - gt = -31.4\left(\frac{m}{s}\right)$

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{17.3^2 + (-31.4)^2} = 35.9\left(\frac{m}{s}\right)$