

*Week\_10: The Assignment Model*

## 1. Assignment model

The assignment model is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. It is actually a special case of the transportation model in which the workers represent the sources, and the jobs represent the destinations. The supply (demand) amount at each source (destination) exactly equals 1. The cost of "transporting" worker  $i$  to job  $j$  is  $c_{ij}$ . In effect, the assignment model can be solved directly as a regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian method. Although the new solution method appears totally unrelated to the transportation model, the algorithm is actually rooted in the simplex method, just as the transportation model.

## 2. LP Representation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point:  $c_{ij}$

On the other hand, a 0-1 integer variable  $x_{ij}$  is defined as follows

$x_{ij} = 1$  if supply point  $i$  is assigned to meet the demands of demand point  $j$

$x_{ij} = 0$  if supply point  $i$  is not assigned to meet the demands of point  $j$

In this case, the general LP representation of an assignment problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t.

$$\sum_j x_{ij} = 1 \quad (i=1,2, \dots, m) \text{ Supply constraints}$$

$$\sum_i x_{ij} = 1 \quad (j=1,2, \dots, n) \text{ Demand constraints}$$

$$x_{ij} = 0 \text{ or } x_{ij} = 1$$

## 2. The Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each  $x_{ij}$  must be a nonnegative integer that is no larger than 1, so each  $x_{ij}$  must equal 0 or 1.

The assignment problem will be solved by the Hungarian method.

**Step 1:** For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the row.

**Step 2.** For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

**Step 3.** Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Example :

Joe Klyne's three children, John, Karen, and Terri, want to earn some money to take care of personal expenses during a school trip to the local zoo. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door, and washing the family cars. To avoid anticipated sibling competition, he asks them to submit (secret) bids for what they feel is fair pay for each of the three chores. The understanding is that all three children will abide by their father's decision as to who gets which chore. Table 5.32 summarizes the bids received. Based on this information, how should Mr. Klyne assign the chores?

Solution:

General description solution of problem in table below.

Klyne's Assignment Problem			
	Mow	Paint	wash
John	\$15	\$10	\$9
Karen	\$9	\$15	\$10
Terri	\$10	\$12	\$8

Step1:

	Mow	Paint	wash	Row min.
John	15	10	9	P1=9
Karen	9	15	10	P2=9
Terri	10	12	8	P3=8

Step2:

	Mow	Paint	wash	
John	6	1	0	
Karen	0	6	1	
Terri	2	4	0	
Column min.	q1=0	q2=1	q3=0	

Step3:

	Mow	Paint	wash
John	6	<u>0</u>	0
Karen	<u>0</u>	5	1
Terri	2	3	<u>0</u>

The cells with underscored zero entries provide the optimum solution. This means that John gets to paint the garage door, Karen gets to mow the lawn, and Terri gets to wash the family cars. The total cost to Mr. Klyne is  $9 + 10 + 8 = \$27$ . This amount also will always equal  $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) = (9 + 9 + 8) + (0 + 1 + 0) = \$27$ .

Example: (chore of children)

Summarizes model the cost elements of the assignment problem describe in table below:

Assignment model					
	Chore				
child		1	2	3	4
	1	\$1	\$4	\$6	\$3
	2	\$9	\$7	\$10	\$9
	3	\$4	\$5	\$11	\$7
	4	\$8	\$7	\$8	\$5

Step 1:

	Chore					
child		1	2	3	4	Row min.
	1	1	4	6	3	P1=1
	2	9	7	10	9	P2=7
	3	4	5	11	7	P3=4
	4	8	7	8	5	P4=5

Step2:

	Chore					
child		1	2	3	4	
	1	0	3	5	2	
	2	2	0	3	2	
	3	0	1	7	3	
	4	3	2	3	0	
	Column min.	q1=0	q2=0	q3=3	q4=0	

Step3:

		Chore				
		1	2	3	4	
child	1	<b>0</b>	3	2	2	
	2	2	<b>0</b>	<b>0</b>	2	
	3	<b>0</b>	1	4	3	
	4	3	2	<b>0</b>	<b>0</b>	

The locations of the zero entries do not allow assigning unique chores to all the children, here, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns. This obstacle can be accounted for by adding the following step to the procedure outlined.

**Step 2a.** If no feasible assignment (with all zero entries) can be secured from steps 1 and 2,

(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix that will cover *all* the zero entries.

		Chore				
		1	2	3	4	
child	1	<b>0</b>	3	2	2	
	2	2	<b>0</b>	<b>0</b>	2	
	3	<b>0</b>	<u>1</u>	4	3	
	4	3	2	<b>0</b>	<b>0</b>	

(ii) Select the *smallest* uncovered entry, **subtract** it from every uncovered entry, then **add** it to every entry at the intersection of two lines.

Smallest value in cell(3,2)=1 subtract from all cell not shadow and result below;

		Chore				
		1	2	3	4	
child	1	<b>0</b>	2	1	1	
	2	2	<b>0</b>	<b>0</b>	2	
	3	<b>0</b>	<u>0</u>	3	2	
	4	3	2	<b>0</b>	<b>0</b>	

And then add to intersection of two line, here, cell(2,1) and cell(4,1) and result below:

		Chore				
		1	2	3	4	
child	1	<b>0</b>	2	1	1	
	2	<b>3</b>	<b>0</b>	<b>0</b>	<b>2</b>	
	3	<b>0</b>	<u>0</u>	3	2	
	4	<b>4</b>	2	<b>0</b>	<b>0</b>	

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 2a. Otherwise, go to step 3 to determine the optimal assignment.

The optimum solution (shown by the underscored zeros in table below) calls for assigning child 1 to chore 1, child 2 to chore 3, child 3 to chore 2, and child 4 to chore 4. The associated optimal cost is  $1 + 10 + 5 + 5 = \$21$ . The same cost is also determined by summing the  $p$ 's, the %'S, and the entry that was subtracted after the shaded cells were determined-that is,  $(1 + 7 + 4 + 5) + (0 + 0 + 3 + 0) + (1) = \$21$ .

		Chore				
		1	2	3	4	
child	1	<u>0</u>	2	1	1	
	2	3	<b>0</b>	<u>0</u>	2	
	3	<b>0</b>	<u>0</u>	3	2	
	4	4	2	<b>0</b>	<u>0</u>	