7.4 a^x and $\log_a x$

We have defined general exponential functions such as 2^x, 10^x, and π^x. In this section we compute their derivatives and integrals. We also define the general logarithmic functions such as $\log_2 x$, $\log_{10} x$, and $\log_\pi x$, and find their derivatives and integrals as well.

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx} \\ &= a^x \ln a. \end{aligned}$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{1}$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

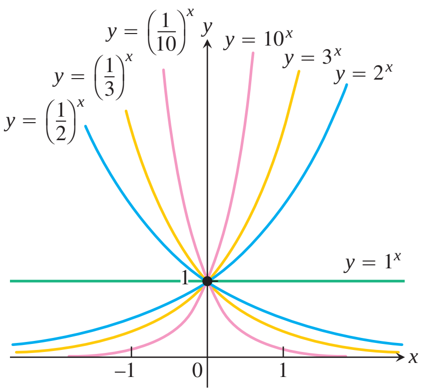

FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

EXAMPLE 1 Differentiating General Exponential Functions

(a) $\dfrac{d}{dx} 3^x = 3^x \ln 3$

(b) $\dfrac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \dfrac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\dfrac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$ ■

From Equation (1), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Figure 7.12).

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 2 Differentiating a General Power Function

Find dy/dx if $y = x^x, \quad x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\
&= e^{x \ln x} \frac{d}{dx}(x \ln x) \\
&= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\
&= x^x (1 + \ln x).
\end{aligned}$$ ■

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Equation (1) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx}\,dx = \int \frac{1}{\ln a}\frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}\int \frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}a^u + C.$$

Writing the first integral in differential form gives

$$\int a^u\,du = \frac{a^u}{\ln a} + C. \tag{2}$$

EXAMPLE 3 Integrating General Exponential Functions

(a) $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ Eq. (2) with $a = 2$, $u = x$

(b) $\displaystyle\int 2^{\sin x}\cos x\,dx$

$\displaystyle = \int 2^u\,du = \frac{2^u}{\ln 2} + C$ $u = \sin x$, $du = \cos x\,dx$, and Eq. (2)

$\displaystyle = \frac{2^{\sin x}}{\ln 2} + C$ u replaced by $\sin x$ ■

Logarithms with Base a

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by $\mathbf{\log_a x}$.

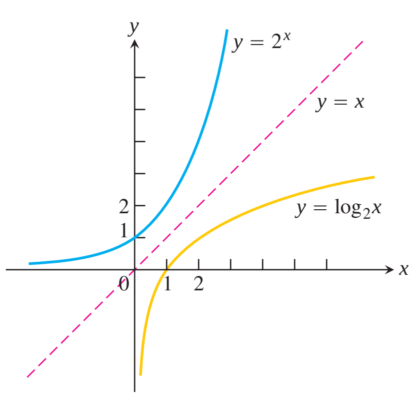

FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line $y = x$ (Figure 7.13). When $a = e$, we have $\log_e x =$ inverse of $e^x = \ln x$. Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{3}$$

$$\log_a(a^x) = x \qquad (\text{all } x) \tag{4}$$

EXAMPLE 4 Applying the Inverse Equations

(a) $\log_2(2^5) = 5$ **(b)** $\log_{10}(10^{-7}) = -7$

(c) $2^{\log_2(3)} = 3$ **(d)** $10^{\log_{10}(4)} = 4$ ■

Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \tag{5}$$

We can derive this equation from Equation (3):

$$\begin{aligned} a^{\log_a(x)} &= x && \text{Eq. (3)} \\ \ln a^{\log_a(x)} &= \ln x && \text{Take the natural logarithm of both sides.} \\ \log_a(x) \cdot \ln a &= \ln x && \text{The Power Rule in Theorem 2} \\ \log_a x &= \frac{\ln x}{\ln a} && \text{Solve for } \log_a x. \end{aligned}$$

For example,

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$

The arithmetic rules satisfied by $\log_a x$ are the same as the ones for $\ln x$ (Theorem 2). These rules, given in Table 7.2, can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\begin{aligned} \ln xy &= \ln x + \ln y && \text{Rule 1 for natural logarithms } \ldots \\ \frac{\ln xy}{\ln a} &= \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} && \ldots \text{ divided by } \ln a \ \ldots \\ \log_a xy &= \log_a x + \log_a y. && \ldots \text{ gives Rule 1 for base } a \text{ logarithms.} \end{aligned}$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
$\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
$\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
$\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
$\log_a x^y = y \log_a x$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

EXAMPLE 5

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$ ■

Base 10 Logarithms

Base 10 logarithms, often called **common logarithms**, appear in many scientific formulas. For example, earthquake intensity is often reported on the logarithmic **Richter scale**. Here the formula is

$$\text{Magnitude } R = \log_{10}\left(\frac{a}{T}\right) + B,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

EXAMPLE 6 Earthquake Intensity

For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ sec, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 6.8 = 7.8.$$

An earthquake of this magnitude can do great damage near its epicenter. ■

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$:

$$\text{pH} = \log_{10}\frac{1}{[H_3O^+]} = -\log_{10}[H_3O^+].$$

Most foods are acidic $(\text{pH} < 7)$.

Food	pH Value
Bananas	4.5–4.7
Grapefruit	3.0–3.3
Oranges	3.0–4.0
Limes	1.8–2.0
Milk	6.3–6.6
Soft drinks	2.0–4.0
Spinach	5.1–5.7

The hydronium ion concentration is measured in moles per liter. Vinegar has a pH of three, distilled water a pH of 7, seawater a pH of 8.15, and household ammonia a pH of 12. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal solution of sodium hydroxide.

Another example of the use of common logarithms is the **decibel** or dB ("dee bee") **scale** for measuring loudness. If I is the **intensity** of sound in watts per square meter, the decibel level of the sound is

$$\textbf{Sound level} = 10\log_{10}(I \times 10^{12})\text{ dB}. \tag{6}$$

Typical sound levels

Threshold of hearing	0 dB
Rustle of leaves	10 dB
Average whisper	20 dB
Quiet automobile	50 dB
Ordinary conversation	65 dB
Pneumatic drill 10 feet away	90 dB
Threshold of pain	120 dB

If you ever wondered why doubling the power of your audio amplifier increases the sound level by only a few decibels, Equation (6) provides the answer. As the following example shows, doubling I adds only about 3 dB.

EXAMPLE 7 Sound Intensity

Doubling I in Equation (6) adds about 3 dB. Writing log for $\log_{10}$ (a common practice), we have

$$\begin{aligned}
\text{Sound level with } I \text{ doubled} &= 10 \log (2I \times 10^{12}) && \text{Eq. (6) with } 2I \text{ for } I \\
&= 10 \log (2 \cdot I \times 10^{12}) \\
&= 10 \log 2 + 10 \log (I \times 10^{12}) \\
&= \text{original sound level} + 10 \log 2 \\
&\approx \text{original sound level} + 3. && \log_{10} 2 \approx 0.30
\end{aligned}$$

EXERCISES 7.4

Algebraic Calculations With a^x and $\log_a x$

Simplify the expressions in Exercises 1–4.

1. a. $5^{\log_5 7}$ b. $8^{\log_8 \sqrt{2}}$ c. $1.3^{\log_{1.3} 75}$
   d. $\log_4 16$ e. $\log_3 \sqrt{3}$ f. $\log_4 \left(\frac{1}{4}\right)$
2. a. $2^{\log_2 3}$ b. $10^{\log_{10}(1/2)}$ c. $\pi^{\log_\pi 7}$
   d. $\log_{11} 121$ e. $\log_{121} 11$ f. $\log_3 \left(\frac{1}{9}\right)$
3. a. $2^{\log_4 x}$ b. $9^{\log_3 x}$ c. $\log_2 \left(e^{(\ln 2)(\sin x)}\right)$
4. a. $25^{\log_5 (3x^2)}$ b. $\log_e (e^x)$ c. $\log_4 \left(2^{e^x \sin x}\right)$

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. a. $\dfrac{\log_2 x}{\log_3 x}$ b. $\dfrac{\log_2 x}{\log_8 x}$ c. $\dfrac{\log_x a}{\log_{x^2} a}$
6. a. $\dfrac{\log_9 x}{\log_3 x}$ b. $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ c. $\dfrac{\log_a b}{\log_b a}$

Solve the equations in Exercises 7–10 for x.

7. $3^{\log_3 (7)} + 2^{\log_2 (5)} = 5^{\log_5 (x)}$
8. $8^{\log_8 (3)} - e^{\ln 5} = x^2 - 7^{\log_7 (3x)}$
9. $3^{\log_3 (x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10} (2)}$
10. $\ln e + 4^{-2 \log_4 (x)} = \dfrac{1}{x} \log_{10} (100)$

Derivatives

In Exercises 11–38, find the derivative of y with respect to the given independent variable.

11. $y = 2^x$
12. $y = 3^{-x}$
13. $y = 5^{\sqrt{s}}$
14. $y = 2^{(s^2)}$
15. $y = x^\pi$
16. $y = t^{1-e}$
17. $y = (\cos \theta)^{\sqrt{2}}$
18. $y = (\ln \theta)^\pi$
19. $y = 7^{\sec \theta} \ln 7$
20. $y = 3^{\tan \theta} \ln 3$
21. $y = 2^{\sin 3t}$
22. $y = 5^{-\cos 2t}$
23. $y = \log_2 5\theta$
24. $y = \log_3(1 + \theta \ln 3)$
25. $y = \log_4 x + \log_4 x^2$
26. $y = \log_{25} e^x - \log_5 \sqrt{x}$
27. $y = \log_2 r \cdot \log_4 r$
28. $y = \log_3 r \cdot \log_9 r$
29. $y = \log_3 \left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$
30. $y = \log_5 \sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$
31. $y = \theta \sin (\log_7 \theta)$
32. $y = \log_7 \left(\dfrac{\sin \theta \cos \theta}{e^\theta 2^\theta}\right)$
33. $y = \log_5 e^x$
34. $y = \log_2 \left(\dfrac{x^2 e^2}{2\sqrt{x+1}}\right)$
35. $y = 3^{\log_2 t}$
36. $y = 3 \log_8 (\log_2 t)$
37. $y = \log_2 (8t^{\ln 2})$
38. $y = t \log_3 \left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

39. $y = (x + 1)^x$
40. $y = x^{(x+1)}$
41. $y = (\sqrt{t})^t$
42. $y = t^{\sqrt{t}}$
43. $y = (\sin x)^x$
44. $y = x^{\sin x}$
45. $y = x^{\ln x}$
46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercises 47–56.

47. $\displaystyle\int 5^x \, dx$
48. $\displaystyle\int (1.3)^x \, dx$

49. $\displaystyle\int_0^1 2^{-\theta}\,d\theta$

50. $\displaystyle\int_{-2}^0 5^{-\theta}\,d\theta$

51. $\displaystyle\int_1^{\sqrt{2}} x2^{(x^2)}\,dx$

52. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

53. $\displaystyle\int_0^{\pi/2} 7^{\cos t}\sin t\,dt$

54. $\displaystyle\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$

55. $\displaystyle\int_2^4 x^{2x}(1+\ln x)\,dx$

56. $\displaystyle\int_1^2 \frac{2^{\ln x}}{x}\,dx$

Evaluate the integrals in Exercises 57–60.

57. $\displaystyle\int 3x^{\sqrt{3}}\,dx$

58. $\displaystyle\int x^{\sqrt{2}-1}\,dx$

59. $\displaystyle\int_0^3 (\sqrt{2}+1)x^{\sqrt{2}}\,dx$

60. $\displaystyle\int_1^e x^{(\ln 2)-1}\,dx$

Evaluate the integrals in Exercises 61–70.

61. $\displaystyle\int \frac{\log_{10} x}{x}\,dx$

62. $\displaystyle\int_1^4 \frac{\log_2 x}{x}\,dx$

63. $\displaystyle\int_1^4 \frac{\ln 2\log_2 x}{x}\,dx$

64. $\displaystyle\int_1^e \frac{2\ln 10\log_{10} x}{x}\,dx$

65. $\displaystyle\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx$

66. $\displaystyle\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\,dx$

67. $\displaystyle\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx$

68. $\displaystyle\int_2^3 \frac{2\log_2(x-1)}{x-1}\,dx$

69. $\displaystyle\int \frac{dx}{x\log_{10} x}$

70. $\displaystyle\int \frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 71–74.

71. $\displaystyle\int_1^{\ln x}\frac{1}{t}\,dt,\quad x>1$

72. $\displaystyle\int_1^{e^x}\frac{1}{t}\,dt$

73. $\displaystyle\int_1^{1/x}\frac{1}{t}\,dt,\quad x>0$

74. $\displaystyle\frac{1}{\ln a}\int_1^x \frac{1}{t}\,dt,\quad x>0$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1+x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.

76. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the x-axis.

77. Blood pH The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for $[H_3O^+]$.

78. Brain fluid pH The cerebrospinal fluid in the brain has a hydronium ion concentration of about $[H_3O^+] = 4.8 \times 10^{-8}$ moles per liter. What is the pH?

79. Audio amplifiers By what factor k do you have to multiply the intensity of I of the sound from your audio amplifier to add 10 dB to the sound level?

80. Audio amplifiers You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration $[H_3O^+]$ (moles/L) and the hydroxyl ion concentration $[OH^-]$ (moles/L) is about 10^{-14}.

a. What value of $[H_3O^+]$ minimizes the sum of the concentrations, $S = [H_3O^+] + [OH^-]$? (*Hint:* Change notation. Let $x = [H_3O^+]$.)

b. What is the pH of a solution in which S has this minimum value?

c. What ratio of $[H_3O^+]$ to $[OH^-]$ minimizes S?

82. Could $\log_a b$ possibly equal $1/\log_b a$? Give reasons for your answer.

T **83.** The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

T **84.** Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. The linearization of 2^x

a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

86. The linearization of $\log_3 x$

a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

Calculations with Other Bases

T **87.** Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the Equation (5), $\log_a x = (\ln x)/(\ln a)$.

Find the following logarithms to five decimal places.

a. $\log_3 8$

b. $\log_7 0.5$

c. $\log_{20} 17$

d. $\log_{0.5} 7$

e. $\ln x$, given that $\log_{10} x = 2.3$

f. $\ln x$, given that $\log_2 x = 1.4$

g. $\ln x$, given that $\log_2 x = -1.5$

h. $\ln x$, given that $\log_{10} x = -0.7$

88. Conversion factors

a. Show that the equation for converting base 10 logarithms to base 2 logarithms is

$$\log_2 x = \frac{\ln 10}{\ln 2}\log_{10} x.$$

b. Show that the equation for converting base a logarithms to base b logarithms is

$$\log_b x = \frac{\ln a}{\ln b}\log_a x.$$