Method of Separation of Variables

In this lecture we discuss the method of separation of variables. It involves a solution which breaks up into a product of two functions each of which contains only one of the variables. The following examples explain this method.

Example 1: Apply the method of separation of variables to solve $u_x = 2u_t + u$ with $u(x,0) = 6e^{-3x}$

Solution : Assume the solution is $u(x,t) = X(x)T(t)$

where X is a function of x alone and T that of t given

$$u_x = X'T \quad \text{and} \quad u_t = XT'$$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \Rightarrow \quad X'T - XT = 2XT'$$

$$(X' - X)T = 2XT' \quad \Rightarrow \quad \frac{X' - X}{2X} = \frac{T'}{T} = k$$

$$\frac{X' - X}{2X} = k \quad \text{and} \quad \frac{T'}{T} = k$$

$$X' = 2Xk + X \quad \text{and} \quad \frac{T'}{T} = k$$

$$\frac{X'}{X} = 2k + 1 \quad \text{and} \quad \frac{T'}{T} = k$$

$$X = ae^{(2k+1)x} \;:\; a = e^{c_1} \quad \text{and} \quad T = be^{kt} \;:\; b = e^{c_2}$$

So $u(x,t) = ae^{(2k+1)x} \times be^{kt} = ce^{(2k+1)x+kt} \;:\; c = ab$

$$u(x,0) = 6e^{-3x} \quad \Rightarrow \quad 6e^{-3x} = ce^{(2k+1)x}$$

$$c = 6 \quad \text{and} \quad 2k + 1 = -3 \quad \Rightarrow \quad k = -2$$

Then $u(x,t) = 6e^{-3x-2t}$

Example 2: Solve by using the method of separation of variables,

$$u_x = 2u_y + u \text{ with } u(x,0) = 3e^{-5x} - 2e^{-3x}$$

Solution : $u = X(x)Y(y) \Rightarrow u_x = X'Y$ and $u_y = XY'$

$$X'Y = 2XY' + XY \Rightarrow X'Y = X(2Y' + Y)$$

$$\frac{X'}{X} = \frac{2Y' + Y}{Y} \Rightarrow \frac{X'}{X} = \frac{2Y'}{Y} + 1 = k$$

$$\frac{X'}{X} = k \text{ and } \frac{2Y'}{Y} + 1 = k \Rightarrow \frac{Y'}{Y} = \frac{k-1}{2}$$

$$X = ae^{kx} \text{ and } Y = be^{\left(\frac{k-1}{2}\right)y}$$

$$u(x,y) = \sum_{n=1}^{2} c_n e^{k_n x} e^{\left(\frac{k_n-1}{2}\right)y} = \sum_{n=1}^{2} c_n e^{k_n x + \left(\frac{k_n-1}{2}\right)y}$$

$$u(x,y) = c_1 e^{k_1 x + \left(\frac{k_1-1}{2}\right)y} + c_2 e^{k_2 x + \left(\frac{k_2-1}{2}\right)y}$$

$$u(x,0) = 3e^{-5x} - 2e^{-3x} \Rightarrow 3e^{-5x} - 2e^{-3x} = c_1 e^{k_1 x} + c_2 e^{k_2 x}$$

So $c_1 = 3$, $k_1 = -5$, $c_2 = -2$ and $k_2 = -3$

Then $u(x,y) = 3e^{-5x-3y} - 2e^{-3x-2y}$

Example 3: Solve $u_x - yu_y = 0$ with $u(0,y) = 2y^3 - 3y^2 + y$

Solution : $u = X(x)Y(y) \Rightarrow u_x = X'Y$ and $u_y = XY'$

$$X'Y - yXY' = 0 \Rightarrow \frac{X'}{X} = \frac{yY'}{Y} = k$$

$$\frac{X'}{X} = k \text{ and } \frac{Y'}{Y} = \frac{k}{y} \Rightarrow X = ae^{kx} \text{ and } Y = by^k$$

$$u(x,y) = \sum_{n=1}^{3} c_n y^{k_n} e^{k_n x} = c_1 y^{k_1} e^{k_1 x} + c_2 y^{k_2} e^{k_2 x} + c_3 y^{k_3} e^{k_3 x}$$

$$u(0,y) = 2y^3 - 3y^2 + y = c_1 y^{k_1} + c_2 y^{k_2} + c_3 y^{k_3}$$

So $c_1 = 2$, $k_1 = 3$, $c_2 = -3$, $k_2 = 2$, $c_2 = 1$ and $k_3 = 1$

Then $u(x,y) = 2y^3 e^{3x} - 3y^2 e^{2x} + ye^x$

Example 4: Solve $u_{xy} + u = 0$ with $u(0,y) = 2\sinh(2y)$

Solution : $u = X(x)Y(y) \quad \Rightarrow \quad u_x = X'Y$ and $u_{xy} = X'Y'$

$$X'Y' + XY = 0 \quad \Rightarrow \quad \frac{X'}{X} = -\frac{Y}{Y'} = k$$

$$\frac{X'}{X} = k \quad \text{and} \quad \frac{Y'}{Y} = -\frac{1}{k} \quad \Rightarrow \quad X = ae^{kx} \quad \text{and} \quad Y = be^{-\frac{y}{k}}$$

$$u(x,y) = ce^{kx-\frac{y}{k}}$$

$$u(0,y) = 4\sinh(2y) \quad \Rightarrow \quad u(0,y) = e^{2y} - e^{-2y}$$

$$u(x,y) = c_1e^{k_1x-\frac{y}{k_1}} + c_2e^{k_2x-\frac{y}{k_2}}$$

$$e^{2y} - e^{-2y} = c_1e^{-\frac{y}{k_1}} + c_2e^{-\frac{y}{k_2}}$$

So $c_1 = 1$, $k_1 = -\frac{1}{2}$, $c_2 = -1$ and $k_2 = \frac{1}{2}$

$$u(x,y) = e^{-\frac{1}{2}x+y} - e^{\frac{1}{2}x-y} = \sinh\left(y - \frac{x}{2}\right)$$

H.W: Apply the method of separation of variables to solve the PDE

1. $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$ $\quad$ *Ans.* $u(x,y) = 4e^{-x-1.5y}$
2. $2u_x - 3u_y = 0$ with $u(x,0) = 5e^{3x}$ $\quad$ *Ans.* $u(x,y) = 5e^{3x+2y}$
3. $yu_x - xu_y = 0$ with $u(x,0) = e^{-x^2}$ $\quad$ *Ans.* $u(x,y) = e^{-(x^2+y^2)}$
4. $u_{xy} - u = 0$ with $u(x,0) = \cosh x$ $\quad$ *Ans.* $u(x,y) = \cosh(x+y)$