1.8 Types of topological vector spaces In the following definitions, X always denotes a topological vector space, with topology τ.

(*a*) X is *locally convex* if there is a local base $\mathscr{B}$ whose members are convex.

(*b*) X is *locally bounded* if 0 has a bounded neighborhood.

(*c*) X is *locally compact* if 0 has a neighborhood whose closure is compact.

(*d*) X is *metrizable* if τ is compatible with some metric d.

(*e*) X is an *F-space* if its topology τ is induced by a complete invariant metric d. (Compare Section 1.25.)

(*f*) X is a *Fréchet space* if X is a locally convex F-space.

(*g*) X is *normable* if a norm exists on X such that the metric induced by the norm is compatible with τ.

(*h*) *Normed spaces* and *Banach spaces* have already been defined (Section 1.2).

(*i*) X has the *Heine-Borel property* if every closed and bounded subset of X is compact.

The terminology of (*e*) and (*f*) is not universally agreed upon: In some texts, local convexity is omitted from the definition of a Fréchet space, whereas others use F-space to describe what we have called Fréchet space.

1.9 Here is a list of some relations between these properties of a topological vector space X.

(*a*) If X is locally bounded, then X has a countable local base [part (*c*) of Theorem 1.15].

(*b*) X is metrizable if and only if X has a countable local base (Theorem 1.24).

(*c*) X is normable if and only if X is locally convex and locally bounded (Theorem 1.39).

(*d*) X has finite dimension if and only if X is locally compact (Theorems 1.21, 1.22).

(*e*) If a locally bounded space X has the Heine-Borel property, then X has finite dimension (Theorem 1.23).

1.13 Theorem *Let X be a topological vector space.*

(*a*) *If $A \subset X$ then $\bar{A} = \bigcap (A + V)$, where V runs through all neighborhoods of* 0.

(*b*) *If $A \subset X$ and $B \subset X$, then $\bar{A} + \bar{B} \subset \overline{A + B}$.*

(*c*) *If Y is a subspace of X, so is $\bar{Y}$.*

(*d*) *If C is a convex subset of X, so are $\bar{C}$ and C°.*

(*e*) *If B is a balanced subset of X, so is $\bar{B}$; if also $0 \in B^{\circ}$ then B° is balanced.*

(*f*) *If E is a bounded subset of X, so is $\bar{E}$.*

PROOF. (*a*) $x \in \bar{A}$ if and only if $(x + V) \cap A \neq \varnothing$ for every neighborhood V of 0, and this happens if and only if $x \in A - V$ for every such V. Since $-V$ is a neighborhood of 0 if and only if V is one, the proof is complete.

(*b*) Take $a \in \bar{A}$, $b \in \bar{B}$; let W be a neighborhood of $a + b$. There are neighborhoods W_1 and W_2 of a and b such that $W_1 + W_2 \subset W$. There exist $x \in A \cap W_1$ and $y \in B \cap W_2$, since $a \in \bar{A}$ and $b \in \bar{B}$. Then $x + y$ lies in $(A + B) \cap W$, so that this intersection is not empty. Consequently, $a + b \in \overline{A + B}$.

(*c*) Suppose α and β are scalars. By the proposition in Section 1.7, $a\bar{Y} = \overline{\alpha Y}$ if $\alpha \neq 0$; if $\alpha = 0$, these two sets are obviously equal. Hence it follows from (*b*) that

$$\alpha \bar{Y} + \beta \bar{Y} = \overline{\alpha Y} + \overline{\beta Y} \subset \overline{\alpha Y + \beta Y} \subset \bar{Y};$$

the assumption that Y is a subspace was used in the last inclusion.

The proofs that convex sets have convex closures and that balanced sets have balanced closures are so similar to this proof of (*c*) that we shall omit them from (*d*) and (*e*).

1.14 Theorem *In a topological vector space X,*

(*a*) *every neighborhood of* 0 *contains a balanced neighborhood of* 0, *and*

(*b*) *every convex neighborhood of* 0 *contains a balanced convex neighborhood of* 0.

PROOF. (*a*) Suppose U is a neighborhood of 0 in X. Since scalar multiplication is continuous, there is a $\delta > 0$ and there is a neighborhood V of 0 in X such that $\alpha V \subset U$ whenever $|\alpha| < \delta$. Let W be the union of all these sets αV. Then W is a neighborhood of 0, W is balanced, and $W \subset U$.

Corollary

(*a*) *Every topological vector space has a balanced local base.*

(*b*) *Every locally convex space has a balanced convex local base.*

Suppose now that X and Y are vector spaces *over the same scalar field*. A mapping $\Lambda: X \to Y$ is said to be *linear* if

$$\Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$$

for all x and y in X and all scalars α and β. Note that one often writes Λx, rather than $\Lambda(x)$, when Λ is linear.

Linear mappings of X into its scalar field are called *linear functionals*.

Here are some properties of linear mappings $\Lambda: X \to Y$ whose proofs are so easy that we omit them; it is assumed that $A \subset X$ and $B \subset Y$:

(*a*) $\Lambda 0 = 0$.

(*b*) If A is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda(A)$.

(*c*) If B is a subspace (or a convex set, or a balanced set) the same is true of $\Lambda^{-1}(B)$.

(*d*) In particular, the set

$$\Lambda^{-1}(\{0\}) = \{x \in X : \Lambda x = 0\} = \mathscr{N}(\Lambda)$$

is a subspace of X, called the *null space* of Λ.

1.18 Theorem *Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some $x \in X$. Then each of the following four properties implies the other three:*

(*a*) *Λ is continuous.*

(*b*) *The null space $\mathscr{N}(\Lambda)$ is closed.*

(*c*) *$\mathscr{N}(\Lambda)$ is not dense in X.*

(*d*) *Λ is bounded in some neighborhood V of 0.*

PROOF. Since $\mathscr{N}(\Lambda) = \Lambda^{-1}(\{0\})$ and $\{0\}$ is a closed subset of the scalar field Φ, (*a*) implies (*b*). By hypothesis, $\mathscr{N}(\Lambda) \neq X$. Hence (*b*) implies (*c*).

Assume (*c*) holds; i.e., assume that the complement of $\mathscr{N}(\Lambda)$ has nonempty interior. By Theorem 1.14,

$$(x + V) \cap \mathscr{N}(\Lambda) = \varnothing \tag{1}$$

for some $x \in X$ and some balanced neighborhood V of 0. Then ΛV is a balanced subset of the field Φ. Thus either ΛV is bounded, in which case (*d*) holds, or $\Lambda V = \Phi$. In the latter case, there exists $y \in V$ such that $\Lambda y = -\Lambda x$, and so $x + y \in \mathscr{N}(\Lambda)$, in contradiction to (1). Thus

Finite-Dimensional Spaces

1.19 Among the simplest Banach spaces are R^n and C^n, the standard n-dimensional vector spaces over R and C, respectively, normed by mean of the usual euclidean metric: If, for example,

$$z = (z_1, \ldots, z_n) \qquad (z_i \in C)$$

is a vector in C^n, then

$$\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

Other norms can be defined on C^n. For example,

$$\|z\| = |z_1| + \cdots + |z_n| \qquad \text{or} \qquad \|z\| = \max\,(|z_i| : 1 \leq i \leq n).$$

These norms correspond, of course, to different metrics on C^n (when $n > 1$) but one can see very easily that they all induce the same topology on C^n. Actually, more is true.

If X is a topological vector space over $\mathcal{C}$, and $\dim X = n$, then every basis of X induces an isomorphism of X onto $\mathcal{C}^n$. Theorem 1.21 will prove that this *isomorphism must be a homeomorphism.* In other words, this says that *the topology of $\mathcal{C}^n$ is the only vector topology that an n-dimensional complex topological vector space can have.*

We shall also see that finite-dimensional subspaces are always closed and that no infinite-dimensional topological vector space is locally compact.

Everything in the preceding discussion remains true with real scalars in place of complex ones.

1.20 Lemma *If X is a complex topological vector space and $f: \mathcal{C}^n \to X$ is linear, then f is continuous.*

PROOF. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathcal{C}^n$: The kth coordinate of e_k is 1, the others are 0. Put $u_k = f(e_k)$, for $k = 1, \ldots, n$. Then $f(z) = z_1 u_1 + \cdots + z_n u_n$ for every $z = (z_1, \ldots, z_n)$ in $\mathcal{C}^n$. Every z_k is a continuous function of z. The continuity of f is therefore an immediate consequence of the fact that addition and scalar multiplication are continuous in X. ////

1.21 Theorem *If n is a positive integer and Y is an n-dimensional subspace of a complex topological vector space X, then*

(*a*) *every isomorphism of $\mathcal{C}^n$ onto Y is a homeomorphism, and*

(*b*) *Y is closed.*

1.22 Theorem *Every locally compact topological vector space X has finite dimension.*

PROOF. The origin of X has a neighborhood V whose closure is compact. By Theorem 1.15, V is bounded, and the sets $2^{-n}V$ ($n = 1, 2, 3, \ldots$) form a local base for X.

The compactness of $\bar{V}$ shows that there exist $x_1, \ldots, x_m$ in X such that

$$\bar{V} \subset (x_1 + \tfrac{1}{2}V) \cup \cdots \cup (x_m + \tfrac{1}{2}V).$$

Let Y be the vector space spanned by $x_1, \ldots, x_m$. Then $\dim Y \leq m$. By Theorem 1.21, Y is a *closed* subspace of X.

Since $V \subset Y + \frac{1}{2}V$ and since $\lambda Y = Y$ for every scalar $\lambda \neq 0$, it follows that

$$\tfrac{1}{2}V \subset Y + \tfrac{1}{4}V$$

so that

$$V \subset Y + \tfrac{1}{2}V \subset Y + Y + \tfrac{1}{4}V = Y + \tfrac{1}{4}V.$$

If we continue in this way, we see that

$$V \subset \bigcap_{n=1}^{\infty} (Y + 2^{-n}V).$$

Since $\{2^{-n}V\}$ is a local base, it now follows from (*a*) of Theorem 1.13 that $V \subset \bar{Y}$. But $\bar{Y} = Y$. Thus $V \subset Y$, which implies that $kV \subset Y$ for $k = 1, 2, 3, \ldots$. Hence $Y = X$, by (*a*) of Theorem 1.15, and consequently $\dim X \leq m$. ////

1.23 Theorem *If X is a locally bounded topological vector space with the Heine-Borel property, then X has finite dimension.*

PROOF. By assumption, the origin of X has a bounded neighborhood V. Statement (*f*) of Theorem 1.13 shows that $\bar{V}$ is also bounded. Thus $\bar{V}$ is compact, by the Heine-Borel property. This says that X is locally compact, hence finite-dimensional, by Theorem 1.22.

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