1.1. Linear spaces and topology

A topological vector space is a linear (or vector) space equipped with a topology which agrees with the linear structure. So we first recall separately basic concepts related to linear spaces and topological spaces. Let $\mathbb{K}$ be an algebraic field (throughout we deal with the field $\mathbb{R}$ of real numbers or with the field $\mathbb{C}$ of complex numbers; so a reader not acquainted with the general notion of an algebraic field can safely ignore it). A set E is called a *linear* (or *vector*) space over the field $\mathbb{K}$ if the elements of E (called vectors) can be added and multiplied by the elements of $\mathbb{K}$, i.e., we are given mappings

$$E\times E \to E,\ (u,v)\mapsto u+v, \quad \mathbb{K}\times E\to E,\ (\lambda,v)\mapsto \lambda v$$

satisfying the following conditions:

(i) $u+v=v+u$ for all $u,v\in E$,

(ii) there is a unique element $0\in E$ (the zero element) for which $v+0=v$ for all $v\in E$,

(iii) for every $v\in E$ there is a unique element $-v$ for which $v+(-v)=0$,

(iv) $\lambda(u+v)=\lambda u+\lambda v$, $\lambda(\mu v)=(\lambda\mu)v$, $(\lambda+\mu)u=\lambda u+\mu u$ and $0v=\lambda 0=0$ for all $u,v\in E$ and $\lambda,\mu\in\mathbb{K}$.

Below we often omit the explicit mentioning the field $\mathbb{K}$ and its elements will be called scalars, and in case of $\mathbb{K}\subset\mathbb{C}$ they will be called numbers. About general fields, see Kurosh [**306**].

1.1.1. Example. Let $\mathbb{K}=\mathbb{R}$ and let T be a nonempty set. Let $\mathbb{R}^T$ be the set of all real functions on T, where the linear operations are defined pointwise:

$$(f+g)(t):=f(t)+g(t), \quad (\lambda f)(t):=\lambda f(t).$$

Then $\mathbb{R}^T$ is a linear space; it is called the product of T copies of the real line or a *power of the real line*.

A *seminorm* on a vector space E is a function $p\colon\ E\to[0,\infty)$ such that

(1) $p(kx)=|k|p(x)\ \ \forall\, k\in\mathbb{K},\ x\in E$;

(2) $p(x_1+x_2)\leqslant p(x_1)+p(x_2)\ \ \forall\, x_1\in E,\ x_2\in E$.

A seminorm p is called a *norm* if $p(x)>0$ whenever $x\neq 0$. For example, $\|x\|=(x,x)^{1/2}$ is a norm on a *Euclidean* space E with an inner product $(\,\cdot\,,\,\cdot\,)$, i.e., $x\mapsto(x,y)$ is linear, $(x,y)\equiv(y,x)$ for real E and $(x,y)\equiv\overline{(y,x)}$ for complex E, $(x,x)\geqslant 0$ and $(x,x)=0$ only for $x=0$.

Two norms p and q are called *equivalent* if for some numbers $c_1,c_2>0$ and all x the inequality $c_1p(x)\leqslant q(x)\leqslant c_2p(x)$ holds.

Let E and F be two vector spaces over the same field. A mapping $A\colon\ E \to F$ is called *linear* (or a *linear operator*) if

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v)$$

for all vectors $u, v \in E$ and all scalars λ, μ. A linear mapping with values in the field of scalars is called a *linear functional*.

The set $\mathrm{Ker}\, A := A^{-1}(0)$ is called the *kernel* of the linear mapping A and the set $\mathrm{Ran}\, A := A(E)$ is called the *range* of A.

For every vector space E, the symbol E^* denotes the vector space of all linear functions on E; it is called the *algebraic dual* to E. The algebraic dual space should not be confused with the *topological dual* considered below and consisting of all continuous linear functions. For the general theory and applications, the topological dual spaces are most important, but the algebraic dual is useful for some examples and constructions.

1.1.2. Definition. *A set V in a real or complex vector space is called convex if $tu + (1-t)v \in V$ for all $u, v \in V$ and $t \in [0, 1]$.*

In other words, a set is convex if along with every two its points it contains the interval joining them. The interval $[a, b]$ with the endpoints a and b is defined by the equality

$$[a, b] := \{x\colon\ x = ta + (1-t)b, t \in [0, 1]\}.$$

Set also

$$(a, b) := [a, b]\backslash\{a, b\}, \quad [a, b) := [a, b]\backslash\{b\}, \quad (a, b] := [a, b]\backslash\{a\}.$$

The *convex hull* (or *convex envelope*) of a nonempty set A in a real or complex vector space E is the intersection $\mathrm{conv}\, A$ of all convex sets containing A.

Thus, the convex hull of the set A is the smallest convex sets containing A. It is readily verified that it consists of all possible sums of the form $t_1 a_1 + \cdots + t_n a_n$, where $a_i \in A$, $t_i \geqslant 0$, $t_1 + \cdots + t_n = 1$.

1.1.3. Definition. *A set M is called circled or balanced if $\lambda x \in M$ for all $x \in M$ and $|\lambda| \leqslant 1$.*

A convex circled set is also called absolutely convex.

The *circled* and *convex circled* (or *absolutely convex*) hulls of a set A in a linear space are, respectively, the smallest circled set and the smallest convex circled set $\mathrm{abs\,conv}\, A$ containing A.

1.1.4. Definition. *If A and B are sets in a linear space E, then we say that A absorbs B (or that the set B is absorbed by the set A) if there exists a number $r > 0$ such that $kB \subset A$ whenever $|k| < r$, $k \in \mathbb{K}$.*

A set in E is called absorbing (or absorbent) if it absorbs every singleton (and then every finite set) in E.

A simple example of a set which does not absorb itself is $\mathbb{K} \setminus \{0\}$; every balanced set absorbs itself (take $r = 1$). If a normed field $\mathbb{K}$ is discrete, then the property to absorb, although formally meaningful, has no useful content since then $\{0\}$ absorbs every set with $r = 1$.

For nonempty sets A and B in a vector space and a scalar λ, we set

$$A + B := \{a + b\colon\ a \in A, b \in B\}, \quad \lambda A := \{\lambda a\colon\ a \in A\};$$

$A + B$ is the *algebraic* (*vector*) *sum* of sets. Further,

$$A - B = A + (-B) = \{a - b\colon\ a \in A, b \in B\}.$$

A *topology* on a set X is a family τ of subsets of this set possessing the following properties:

(i) $X, \varnothing \in \tau$;

(ii) if $V_1, V_2 \in \tau$, then $V_1 \cap V_2 \in \tau$;

(iii) the union of every collection of sets from τ belongs to τ.

Hence the minimal topology is $(X, \varnothing)$ and the maximal topology is 2^X, the class of all subsets of X.

A *topological space* is a pair (X, τ), where X is a set, called the set of elements of this topological space, and τ is a topology on X. The elements of τ are called *open subsets* of the topological space X.

A subset of a topological space is called *closed* if its complement is open. The topology in X can be also defined by introducing the class $\mathcal{F}$ of all closed sets, which must satisfy the following conditions:

(i) $X, \varnothing \in \mathcal{F}$;

(ii) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$;

(iii) the intersection of every collection of sets from $\mathcal{F}$ belongs to $\mathcal{F}$.

An important subclass of the class of topological spaces is formed by metric spaces. Although a grasp knowledge of them is assumed, we recall that a *metric space* (M, d) is a pair, where M is a set and $d\colon\ M \times M \to [0, +\infty)$ is a function, called a *metric*, that satisfies the following conditions:

(i) $d(a, b) = d(b, a)$, in addition, $d(a, b) = 0$ if and only if $a = b$,

(ii) $d(a, c) \leqslant d(a, b) + d(b, c)$ (*the triangle inequality*).

A linear space with a norm $\|\cdot\|$ (a *normed space*) is a metric space with the metric $d(x, y) = \|x - y\|$.

The topological space is called *metrizable* if its topology is obtained in the indicated way from some metric on it. Different metrics can generate the same topology. For example, the usual metric on the real line generates the same topology as the bounded metric $d(x, y) = \min(1, |x - y|)$. Below we encounter many examples of nonmetrizable spaces, so we do not give artificial examples of this sort. The *discrete topology* on X is $\tau = 2^X$.

We assume that the concept of a complete metric space is known (anyway, we recall it in § 1.7).

A *pseudometric* on a set M is a function $\varrho\colon M \times M \to [0, \infty)$ with the following properties:

(1) $\varrho(x, x) = 0$;

(2) $\varrho(x, y) = \varrho(y, x)$;

(3) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$.

If the triangle inequality (3) is written as

(3′) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(y, z)$,

then conditions (2) and (3) together will be equivalent to the pair of conditions (2) and (3′), but (2) will follow from (1) and (3′) by replacing z in (3′) with x.

The pseudometric ϱ on a nonempty set M generates a topology on this set in the same way as a metric: a set $V \subset M$ is called open in the topology generated by the pseudometric ϱ if for each $x \in V$ there is $\varepsilon > 0$ such that $\{z\colon\ \varrho(z, x) < \varepsilon\}$ is contained in V. In addition, any pseudometric generates a metric on the set of equivalence classes if we set $x \sim y$ when $d(x, y) = 0$.

An *open neighborhood of a point* x is any open set containing x. Sometimes it is useful to employ a broader concept of a *neighborhood* of a point (not necessarily open!) as an arbitrary set containing some open neighborhood of this point. For example, it becomes possible to speak of closed neighborhoods in this sense.

A *base of the topology* (*topology base*) is any collection of open sets with the property that all possible unions of the elements of this collection give all nonempty open sets.

A *base of the topology at a point* x or a *fundamental system of neighborhoods of the point* x is any collection of open neighborhoods of the point x with the property that every neighborhood of x contains an element of this collection. Sometimes, similarly to neighborhoods, bases of not necessarily open neighborhoods are used. A *prebase* of neighborhoods of a point in a topological space is a family of neighborhoods of this point finite intersections of elements of which form a base of its neighborhoods.

1.2.1. Definition. *A topological vector space over a topological field $\mathbb{K}$ is a vector space E over $\mathbb{K}$ equipped with a topology with respect to which the following two mappings are continuous, where $E\times E$ and $\mathbb{K}\times E$ are equipped with the products of the corresponding topologies*: 1) $(x_1, x_2) \mapsto x_1 + x_2,\ E\times E \to E$ (*addition of vectors*), 2) $(k, x) \mapsto kx,\ \mathbb{K}\times E \to E$ (*multiplication of vectors by scalars*).

Such a topology on E is called *compatible with the vector space structure* (or we say that it *agrees with the vector space structure*). A topological vector space E with the topology τ is denoted by the symbol (E, τ). We observe that in the definition of a topological field one requires the same conditions with $\mathbb{K}$ in place of E and the continuity of $k \mapsto k^{-1}$ outside of zero.

Two topological vector spaces over the same field are called *isomorphic* if there exists a continuous linear one-to-one mapping of one of the two spaces onto the other such that the inverse mapping is also continuous (i.e., a linear homeomorphism). The *dimension of a topological vector space* (E, τ) is the dimension of the vector space E.

The continuity of the mapping 1) implies that the topology of any topological vector space (E, τ) is invariant with respect to translations (i.e., for every $a \in E$ the mapping $x \mapsto x + a$ is a homeomorphism of E); hence the topology of a topological vector space can be reconstructed if we know a fundamental system of neighborhoods of zero.

If $\mathcal{U}$ is a base of neighborhoods of zero and $a \in E$, then the collection of sets of the form $a + V$, where $V \in \mathcal{U}$, is a base of neighborhoods of the point a. Thus, for defining a topology of a topological vector space it suffices to define a base of neighborhoods of zero; this is usually done in most of applications of the theory of topological vector spaces. However, not every system of subsets of a vector space can serve as a base of neighborhoods of zero of a topology compatible with the vector space structure; conditions sufficient for this are indicated in Proposition 1.2.7.

1.2.2. Proposition. (a) *Every base of neighborhoods of zero $\mathcal{U}$ in a topological vector space has the following properties*:

(1) *for every $V \in \mathcal{U}$ there exists a set $W \in \mathcal{U}$ such that $W + W \subset V$*;

(2) *every $V \in \mathcal{U}$ is an absorbent set.*

(b) *In every topological vector space there exists a base of neighborhoods of zero $\mathcal{U}_0$ having also the following properties*:

(3) *every $V \in \mathcal{U}_0$ is a circled closed set*;

(4) *if $V \in \mathcal{U}_0$, then $kV \in \mathcal{U}_0$ for every $k \in \mathbb{K}$, $k \neq 0$.*

1.2.4. Corollary. *Each point in a topological vector space possesses a base of neighborhoods consisting of closed sets* (*i.e., every topological vector space is a regular topological space, as well as any topological group*).

PROOF. Indeed, if $\mathcal{U}$ is a base of closed neighborhoods of zero, then $a + \mathcal{U}$ is a base of closed neighborhoods of the point a for every a. □

1.2.5. Corollary. *A topological vector space is a T_3-space* (*hence Hausdorff*) *if and only if it is a T_0-space.*

Actually, more is true: a Hausdorff topological vector space is completely regular, which will be established in § 1.6.

1.2.6. Corollary. *A topological vector space is Hausdorff if and only if the intersection of all its neighborhoods of zero is the zero element of this space.*

It is clear that any topological vector space is Hausdorff space.

References:

1. Robertson, W. Topological vector spaces. CUP Archive, Vol. 53, 1980.
2. Bourbaki, N. Topological vector spaces: Chapters 1–5. Springer Science and Business Media, 2013.
3. Bogachev, V. I., Smolyanov, O. G., Sobolev. Topological vector spaces and their applications. Springer, Vol. 1, 2017.
4. Rudin, W. Functional analysis, mir, 1975.