Metrization

We recall that a topology τ on a set X is said to be *metrizable* if there is a metric d on X which is compatible with τ. In that case, the balls with radius $1/n$ centered at x form a local base at x. This gives a necessary condition for metrizability which, for topological vector spaces, turns out to be also sufficient.

1.24 Theorem *If X is a topological vector space with a countable local base, then there is a metric d on X such that*

(*a*) *d is compatible with the topology of X,*

(*b*) *the open balls centered at 0 are balanced, and*

(*c*) *d is invariant: $d(x + z, y + z) = d(x, y)$ for $x, y, z \in X$.*

If, in addition, X is locally convex, then d can be chosen so as to satisfy (*a*), (*b*), (*c*), *and also*

(*d*) *all open balls are convex.*

1.25 Cauchy sequences (*a*) Suppose d is a metric on a set X. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if to every $\varepsilon > 0$ there corresponds an integer N such that $d(x_m, x_n) < \varepsilon$ whenever $m > N$ and $n > N$. If every Cauchy sequence in X converges to a point of X, then d is said to be a *complete* metric on X.

(*b*) Let τ be the topology of a topological vector space X. The notion of Cauchy sequence can be defined in this setting without reference to any metric: Fix a local base $\mathscr{B}$ for τ. A sequence $\{x_n\}$ in X is then said to be a *Cauchy sequence* if to every $V \in \mathscr{B}$ corresponds an N such that $x_n - x_m \in V$ if $n > N$ and $m > N$.

It is clear that different local bases for the same τ give rise to the same class of Cauchy sequences.

(*c*) Suppose now that X is a topological vector space whose topology τ is compatible with an *invariant* metric d. Let us temporarily use the terms d-Cauchy sequence and τ-Cauchy sequence for the concepts defined in (*a*) and (*b*), respectively. Since

$$d(x_n, x_m) = d(x_n - x_m, 0),$$

and since the d-balls centered at the origin form a local base for τ, we conclude:

A sequence $\{x_n\}$ in X is a d-Cauchy sequence if and only if it is a τ-Cauchy sequence.

Consequently, any two invariant metrics on X that are compatible with τ have the same Cauchy sequences. They clearly also have the same convergent sequences (namely, the τ-convergent ones). These remarks prove the following fact:

If d_1 and d_2 are invariant metrics on a vector space X which induce the same topology on X, then

(*a*) *d_1 and d_2 have the same Cauchy sequences, and*
(*b*) *d_1 is complete if and only if d_2 is complete.*

1.27 Theorem *Suppose Y is a subspace of a topological vector space X, and Y is an F-space (in the topology inherited from X). Then Y is a closed subspace of X.*

PROOF. Choose an invariant metric d on Y, compatible with its topology. Let

$$B_{1/n} = \left\{y \in Y : d(y, 0) < \frac{1}{n}\right\},$$

let U_n be a neighborhood of 0 in X such that $Y \cap U_n = B_{1/n}$, and choose symmetric neighborhoods V_n of 0 in X such that $V_n + V_n \subset U_n$ and $V_{n+1} \subset V_n$.

Suppose $x \in \bar{Y}$, and define

$$E_n = Y \cap (x + V_n) \qquad (n = 1, 2, 3, \ldots).$$

If $y_1 \in E_n$ and $y_2 \in E_n$, then $y_1 - y_2$ lies in Y and also in $V_n + V_n \subset U_n$, hence in $B_{1/n}$. The diameters of the sets E_n therefore tend to 0. Since each E_n is nonempty and since Y is complete, it follows that the Y-closures of the sets E_n have exactly one point y_0 in common.

Let W be a neighborhood of 0 in X, and define

$$F_n = Y \cap (x + W \cap V_n).$$

The preceding argument shows that the Y-closures of the sets F_n have one common point y_W. But $F_n \subset E_n$. Hence $y_W = y_0$. Since $F_n \subset x + W$, it follows that y_0 lies in the X-closure of $x + W$, for every W. This implies $y_0 = x$. Thus $x \in Y$. This proves that $\bar{Y} = Y$. ////

1.28 Theorem

(*a*) *If d is a translation-invariant metric on a vector space X then*

$$d(nx, 0) \leq nd(x, 0)$$

for every $x \in X$ and for $n = 1, 2, 3, \ldots$.

(*b*) *If $\{x_n\}$ is a sequence in a metrizable topological vector space X and if $x_n \to 0$ as $n \to \infty$, then there are positive scalars γ_n such that $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.*

PROOF. Statement (*a*) follows from

$$d(nx, 0) \leq \sum_{k=1}^{n} d(kx, (k-1)x) = nd(x, 0).$$

To prove (*b*), let d be a metric as in (*a*), compatible with the topology of X. Since $d(x_n, 0) \to 0$, there is an increasing sequence of positive integers n_k such that $d(x_n, 0) < k^{-2}$ if $n \geq n_k$. Put $\gamma_n = 1$ if $n < n_1$; put $\gamma_n = k$ if $n_k \leq n < n_{k+1}$. For such n,

$$d(\gamma_n x_n, 0) = d(kx_n, 0) \leq kd(x_n, 0) < k^{-1}.$$

Hence $\gamma_n x_n \to 0$ as $n \to \infty$. ////

1.29 Bounded sets The notion of a *bounded subset of a topological vector space* X was defined in Section 1.6 and has been encountered several times since then. When X is metrizable, there is a possibility of misunderstanding, since another very familiar notion of boundedness exists in metric spaces.

If d is a metric on a set X, a set $E \subset X$ is said to be d-bounded if there is a number $M < \infty$ such that $d(z, y) \leq M$ for all x and y in E.

If X is a topological vector space with a compatible metric d, the bounded sets and the d-bounded ones need not be the same, even if d is invariant. For instance, if d is a metric such as the one constructed in Theorem 1.24, then X itself is d-bounded (with $M = 1$) but, as we shall see presently, X cannot be bounded, unless $X = \{0\}$. If X is a normed space and d is the metric induced by the norm, then the two notions of boundedness coincide; but if d is replaced by $d_1 = d/(1 + d)$ (an invariant metric which induces the same topology) they do not.

Whenever bounded subsets of a topological vector space are discussed, *it will be understood that the definition is as in Section* 1.6: A set E is bounded if, for every neighborhood V of 0, we have $E \subset tV$ for all sufficiently large t.

We already saw (Theorem 1.15) that *compact sets are bounded*. To see another type of example, let us prove that *Cauchy sequences are bounded* (hence *convergent sequences are bounded*): If $\{x_n\}$ is a Cauchy sequence in X, and V and W are balanced neighborhoods of 0 with $V + V \subset W$, then [part (*b*) of Section 1.25] there exists N such that $x_n \in x_N + V$ for all $n \geq N$. Take $s > 1$ so that $x_N \in sV$. Then

$$x_n \in sV + V \subset sV + sV \subset sW \qquad (n \geq N).$$

Hence $x_n \in tW$ for all $n \geq 1$, if t is sufficiently large.

Also, closures of bounded sets are bounded (Theorem 1.13).

On the other hand, if $x \neq 0$ and $E = \{nx\colon n = 1, 2, 3, \ldots\}$, then E is not bounded, because there is a neighborhood V of 0 that does not contain x; hence nx is not in nV; it follows that no nV contains E.

Consequently, *no subspace of* X (*other than* $\{0\}$) *can be bounded.*

The next theorem characterizes boundedness in terms of sequences.

1.30 Theorem *The following two properties of a set E in a topological vector space are equivalent:*

(a) *E is bounded.*

(b) *If $\{x_n\}$ is a sequence in E and $\{\alpha_n\}$ is a sequence of scalars such that $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$.*

PROOF. Suppose E is bounded. Let V be a balanced neighborhood of 0 in X. Then $E \subset tV$ for some t. If $x_n \in E$ and $\alpha_n \to 0$, there exists N such that $|\alpha_n| t < 1$ if $n > N$. Since $t^{-1}E \subset V$ and V is balanced, $\alpha_n x_n \in V$ for all $n > N$. Thus $\alpha_n x_n \to 0$.

Conversely, if E is not bounded, there is a neighborhood V of 0 and a sequence $r_n \to \infty$ such that no $r_n V$ contains E. Choose $x_n \in E$ such that $x_n \notin r_n V$. Then no $r_n^{-1} x_n$ is in V, so that $\{r_n^{-1} x_n\}$ does not converge to 0. ////

1.31 Bounded linear transformations Suppose X and Y are topological vector spaces and $\Lambda: X \to Y$ is linear. Λ is said to be *bounded* if Λ maps bounded sets into bounded sets, i.e., if $\Lambda(E)$ is a bounded subset of Y for every bounded set $E \subset X$.

This definition conflicts with the usual notion of a bounded function as being one whose range is a bounded set. In that sense, no linear function (other than 0) could ever be bounded. Thus when bounded linear mappings (or transformations) are discussed, it is to be understood that the definition is in terms of bounded sets, as above.

1.32 Theorem *Suppose X and Y are topological vector spaces and $\Lambda: X \to Y$ is linear. Among the following four properties of Λ, the implications*

$$(a) \to (b) \to (c)$$

hold. If X is metrizable, then also

$$(c) \to (d) \to (a),$$

so that all four properties are equivalent.

(*a*) *Λ is continuous.*

(*b*) *Λ is bounded.*

(*c*) *If $x_n \to 0$ then $\{\Lambda x_n : n = 1, 2, 3, \ldots\}$ is bounded.*

(*d*) *If $x_n \to 0$ then $\Lambda x_n \to 0$.*