

Taylor-Maclaurin Series

Consider a function f defined by a power series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

with radius of convergence $R > 0$. If we write out the expansion of $f(x)$ as

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \dots$$

we observe that $f(a) = c_0$. Moreover

$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \dots$$

$$f''(x) = 2c_2 + 3 \cdot 2c_3(x - a) + 4 \times 3c_4(x - a)^2 + \dots$$

$$f^{(3)}(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x - a) + \dots$$

After computing the above derivatives we observe that

$$f'(a) = c_1$$

$$f''(a) = 2c_2 \quad \Rightarrow \quad c_2 = \frac{f''(a)}{2!}$$

$$f^{(3)}(a) = 3 \times 2c_3 \quad \Rightarrow \quad c_3 = \frac{f^{(3)}(a)}{3!}$$

In general we have

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Suppose that $f(x)$ has a power series expansion at $x = a$ with radius of convergence $R > 0$, then the series expansion of $f(x)$ takes the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = \sum_{n=0}^{\infty} c_n (x - a)^n$$

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f^{(3)}(a)}{3!} (x - a)^3 + \dots$$

Which is called Taylor series.

If $a = 0$, then

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots$$

Which is called Maclaurin Series.

Example 1: Compute the Maclaurin series of the following functions

$$\begin{aligned} 1. f(x) &= e^x & 2. f(x) &= e^{x^2} \\ f(x) &= e^x \Rightarrow f(0) = e^0 = 1 \\ f'(x) &= e^x \Rightarrow f'(0) = e^0 = 1 \\ f''(x) &= e^x \Rightarrow f''(0) = e^0 = 1 \\ f^{(3)}(x) &= e^x \Rightarrow f^{(3)}(0) = e^0 = 1 \end{aligned}$$

$$\begin{aligned} 1. e^x &= \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ 2. e^{x^2} &= \frac{1}{0!} + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \end{aligned}$$

Example 2: Compute the Maclaurin series of the following functions

$$\begin{aligned} 1. f(x) &= \sin x & 2. f(x) &= \frac{\sin(x^2)}{x^2} \\ f(x) &= \sin x \Rightarrow f(0) = \sin 0 = 0 \\ f'(x) &= \cos x \Rightarrow f'(0) = \cos 0 = 1 \\ f''(x) &= -\sin x \Rightarrow f''(0) = -\sin 0 = 0 \\ f^{(3)}(x) &= -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1 \end{aligned}$$

we note that $f^{(2n+1)}(x) = (-1)^n \cos x \Rightarrow f^{(2n+1)}(0) = (-1)^n$

$$f^{(2n)}(x) = (-1)^n \sin x \Rightarrow f^{(2n)}(0) = 0$$

$$\begin{aligned} 1. \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ 2. \sin(x^2) &= \frac{(x^2)}{1!} - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} \end{aligned}$$

$$\sin(x^2) = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\frac{\sin(x^2)}{x^2} = \frac{1}{1!} - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n+1)!}$$

Taylor polynomials and Maclaurin polynomials.

The partial sums of Taylor (Maclaurin) series are called Taylor (Maclaurin) polynomials. More precisely, the Taylor polynomial of degree k of $f(x)$ at $x = a$ is the polynomial

$$P_k(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n$$

and the Maclaurin polynomial of degree k of $f(x)$ (at $x = 0$) is the polynomial

$$P_k(x) = \sum_{n=0}^k \frac{f^{(n)}(0)}{n!} x^n$$

Example 3: Compute the Maclaurin polynomial of degree 4 for the function

$$f(x) = \cos x \ln(1-x)$$

Maclaurin polynomial $P_4(x)$ of degree 4 of $f(x)$ is

$$P_4(x) = \sum_{n=0}^4 \frac{f^{(n)}(0)}{n!} x^n = \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4$$

$$f_1(x) = \cos x \quad \Rightarrow \quad f_1(0) = 1$$

$$f_1^{(2n+1)}(x) = (-1)^n \sin x \quad \Rightarrow \quad f_1^{(2n+1)}(0) = 0$$

$$f_1^{(2n)}(x) = (-1)^n \cos x \quad \Rightarrow \quad f_1^{(2n)}(0) = (-1)^n$$

$$f_1(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$f_2(x) = \ln(1-x) \quad \Rightarrow \quad f_2(0) = \ln(1) = 0$$

$$f_2'(x) = \frac{-1}{1-x} = -(1-x)^{-1} \quad \Rightarrow \quad f_2'(0) = -1$$

$$f_2''(x) = -(1-x)^{-2} \quad \Rightarrow \quad f_2''(0) = -1$$

$$f_2^{(3)}(x) = -2(1-x)^{-3} \quad \Rightarrow \quad f_2^{(3)}(0) = -2$$

$$f_2(x) = \ln(1-x) = 0 - x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{6x^4}{4!} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$f(x) = \cos x \ln(1-x) = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)$$

$$P_4(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^4}{4} = -x - \frac{x^2}{2} + \frac{x^3}{6}$$

Example3 : Compute the first four terms in the power series expansion of following

$$f(x) = \frac{\ln(1+x)}{(1+x)}$$

$$f_1(x) = \ln(1+x) \quad \Rightarrow \quad f_1(0) = \ln(1) = 0$$

$$f_1'(x) = \frac{1}{(1+x)} = (1+x)^{-1} \quad \Rightarrow \quad f_1'(0) = 1$$

$$f_1''(x) = -(1+x)^{-2} \quad \Rightarrow \quad f_1''(0) = -1$$

$$f_1^{(3)}(x) = 2(1+x)^{-3} \quad \Rightarrow \quad f_1^{(3)}(0) = 2$$

$$f_1^{(4)}(x) = -6(1+x)^{-4} \quad \Rightarrow \quad f_1^{(4)}(0) = -6$$

$$f_1(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$f_2(x) = \frac{1}{(1+x)} = (1+x)^{-1} \quad \Rightarrow \quad f_2(0) = 1$$

$$f_2'(x) = -(1+x)^{-2} \quad \Rightarrow \quad f_2'(0) = -1$$

$$f_2''(x) = 2(1+x)^{-2} \quad \Rightarrow \quad f_2''(0) = 2$$

$$f_2^{(3)}(x) = -6(1+x)^{-3} \quad \Rightarrow \quad f_2^{(3)}(0) = -6$$

$$f_2(x) = \frac{1}{(1+x)} = 1 - x + x^2 - x^3 + x^4$$

$$f(x) = \frac{\ln(1+x)}{(1+x)} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \right) (1 - x + x^2 - x^3 + x^4)$$

$$P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - x^2 + \frac{x^3}{2} - \frac{x^4}{3} + x^3 - \frac{x^4}{2} - x^4$$

$$P_4(x) = x - \frac{3x^2}{2} + \frac{11x^3}{6} - \frac{25x^4}{12}$$

Exercises

Compute the Maclaurin series of the following functions

1. $f(x) = 1/(1-x)$

2. $f(x) = \sqrt{1+x}$

□

Compute the first four terms in the power series expansion of following

3. $f(x) = \sqrt{1+x} \cos x$

4. $f(x) = (\sin x)/(1-x)$ □