Seminorms and Local Convexity

1.33 Definitions A *seminorm* on a vector space X is a real-valued function p on X such that

(*a*) $p(x + y) \le p(x) + p(y)$ and
(*b*) $p(\alpha x) = |\alpha| p(x)$

for all x and y in X and all scalars α.

Property (*a*) is called *subadditivity*. Theorem 1.34 will show that a seminorm p is a norm if it satisfies

(*c*) $p(x) \ne 0$ if $x \ne 0$.

A family $\mathscr{P}$ of seminorms on X is said to be *separating* if to each $x \ne 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \ne 0$.

Next, consider a convex set $A \subset X$ which is *absorbing*, in the sense that every $x \in X$ lies in tA for some $t = t(x) > 0$. [For example, (*a*) of Theorem 1.15 implies that every neighborhood of 0 in a topological vector space is absorbing. Every absorbing set obviously contains 0.] The *Minkowski functional* μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\} \qquad (x \in X).$$

Note that $\mu_A(x) < \infty$ for all $x \in X$, since A is absorbing. The seminorms on X will turn out to be precisely the Minkowski functionals of *balanced* convex absorbing sets.

Seminorms are closely related to local convexity, in two ways: In every locally convex space there exists a separating family of *continuous* seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continuous. This is a frequently used method of introducing a topology. The details are contained in Theorems 1.36 and 1.37.

1.34 Theorem *Suppose p is a seminorm on a vector space X. Then*

(*a*) $p(0) = 0$.

(*b*) $|p(x) - p(y)| \le p(x - y)$.

(*c*) $p(x) \ge 0$.

(*d*) $\{x : p(x) = 0\}$ *is a subspace of* X.

(*e*) *The set* $B = \{x : p(x) < 1\}$ *is convex, balanced, absorbing, and* $p = \mu_B$.

PROOF. Statement (*a*) follows from $p(\alpha x) = |\alpha| p(x)$, with $\alpha = 0$. The subadditivity of p shows that

$$p(x) = p(x - y + y) \le p(x - y) + p(y)$$

so that $p(x) - p(y) \le p(x - y)$. This also holds with x and y interchanged. Since $p(x - y) = p(y - x)$, (*b*) follows. With $y = 0$, (*b*) implies (*c*). If $p(x) = p(y) = 0$ and α, β are scalars, (*c*) implies

$$0 \le p(\alpha x + \beta y) \le |\alpha| p(x) + |\beta| p(y) = 0.$$

This proves (*d*).

As to (*e*), it is clear that B is balanced. If $x \in B$, $y \in B$, and $0 < t < 1$, then

$$p(tx + (1 - t)y) \le tp(x) + (1 - t)p(y) < 1.$$

Thus B is convex. If $x \in X$ and $s > p(x)$ then $p(s^{-1}x) = s^{-1}p(x) < 1$. This shows that B is absorbing and also that $\mu_B(x) \le s$. Hence $\mu_B \le p$. But if $0 < t \le p(x)$ then $p(t^{-1}x) \ge 1$, and so $t^{-1}x$ is not in B. This implies $p(x) \le \mu_B(x)$ and completes the proof. ////

1.35 Theorem *Suppose A is a convex absorbing set in a vector space X. Then*

(*a*) $\mu_A(x + y) \le \mu_A(x) + \mu_A(y)$.

(*b*) $\mu_A(tx) = t\mu_A(x)$ *if* $t \ge 0$.

(*c*) μ_A *is a seminorm if* A *is balanced.*

(*d*) *If* $B = \{x\colon \mu_A(x) < 1\}$ *and* $C = \{x\colon \mu_A(x) \le 1\}$, *then* $B \subset A \subset C$ *and* $\mu_B = \mu_A = \mu_C$.

PROOF. If $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$, for some $\varepsilon > 0$, then x/t and y/s are in A; hence so is their convex combination

$$\frac{x+y}{s+t} = \frac{t}{s+t} \cdot \frac{x}{t} + \frac{s}{s+t} \cdot \frac{y}{s}.$$

This shows that $\mu_A(x+y) \le s + t = \mu_A(x) + \mu_A(y) + 2\varepsilon$, and (*a*) is proved.

Property (*b*) is clear, and (*c*) follows from (*a*) and (*b*).

When we turn to (*d*), the inclusions $B \subset A \subset C$ show that $\mu_C \le \mu_A \le \mu_B$. To prove equality, fix $x \in X$, and choose s, t so that $\mu_C(x) < s < t$. Then $x/s \in C$, $\mu_A(x/s) \le 1$, $\mu_A(x/t) \le s/t < 1$; hence $x/t \in B$, so that $\mu_B(x) \le t$. This holds for every $t > \mu_C(x)$. Hence $\mu_B(x) \le \mu_C(x)$. ////

1.36 Theorem *Suppose $\mathscr{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathscr{B}$ its Minkowski functional μ_V. Then*

(*a*) $V = \{x \in X\colon \mu_V(x) < 1\}$, *for every* $V \in \mathscr{B}$, *and*

(*b*) $\{\mu_V\colon V \in \mathscr{B}\}$ *is a separating family of continuous seminorms on* X.

PROOF. If $x \in V$, then $x/t \in V$ for some $t < 1$, because V is open; hence $\mu_V(x) < 1$. If $x \notin V$, then $x/t \in V$ implies $t \ge 1$, because V is balanced; hence $\mu_V(x) \ge 1$. This proves (*a*).

Theorem 1.35 shows that each μ_V is a seminorm. If $r > 0$, it follows from (*a*) and Theorem 1.34 that

$$|\mu_V(x) - \mu_V(y)| \le \mu_V(x - y) < r$$

if $x - y \in rV$. Hence μ_V is continuous. If $x \in X$ and $x \ne 0$, then $x \notin V$ for some $V \in \mathscr{B}$. For this V, $\mu_V(x) \ge 1$. Thus $\{\mu_V\}$ is separating. ////

1.37 Theorem *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and to each positive integer n the set*

$$V(p, n) = \left\{x \colon p(x) < \frac{1}{n}\right\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

(a) *every $p \in \mathscr{P}$ is continuous, and*

(b) *a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.*

Examples

1.44 The spaces $C(\Omega)$ If Ω is a nonempty open set in some euclidean space, then Ω is the union of countably many compact sets $K_n \neq \varnothing$ which can be chosen so that K_n lies in the interior of K_{n+1} ($n = 1, 2, 3, \ldots$). $C(\Omega)$ is the vector space of all complex-valued continuous functions on Ω, topologized by the separating family of seminorms

(1) $$p_n(f) = \sup\{|f(x)| : x \in K_n\},$$

in accordance with Theorem 1.37. Since $p_1 \leq p_2 \leq \cdots$, the sets

(2) $$V_n = \left\{f \in C(\Omega) \colon p_n(f) < \frac{1}{n}\right\} \qquad (n = 1, 2, 3, \ldots)$$

form a convex local base for $C(\Omega)$. According to remark (c) of Section 1.38, the topology of $C(\Omega)$ is compatible with the metric

(3) $$d(f, g) = \max_n \frac{2^{-n} p_n(f - g)}{1 + p_n(f - g)}.$$

If $\{f_i\}$ is a Cauchy sequence relative to this metric, then $p_n(f_i - f_j) \to 0$ for every n, as $i, j \to \infty$, so that $\{f_i\}$ converges uniformly on K_n, to a function $f \in C(\Omega)$. An easy computation then shows $d(f, f_i) \to 0$. Thus d is a complete metric. We have now proved *that $C(\Omega)$ is a Fréchet space.*

By (*b*) of Theorem 1.37, a set $E \subset C(\Omega)$ is bounded if and only if there are numbers $M_n < \infty$ such that $p_n(f) \le M_n$ for all $f \in E$; explicitly,

$$(4) \qquad |f(x)| \le M_n \qquad \text{if } f \in E \text{ and } x \in K_n.$$

Since every V_n contains an f for which $p_{n+1}(f)$ is as large as we please, it follows that no V_n is bounded. *Thus $C(\Omega)$ is not locally bounded, hence is not normable.*

1.45 The spaces $H(\Omega)$ Let Ω now be a nonempty open subset of the complex plane, define $C(\Omega)$ as in Section 1.44, and let $H(\Omega)$ be the subspace of $C(\Omega)$ that consists of the holomorphic functions in Ω. Since sequences of holomorphic functions that converge uniformly on compact sets have holomorphic limits, $H(\Omega)$ is a closed subspace of $C(\Omega)$. *Hence $H(\Omega)$ is a Fréchet space.*

We shall now prove that $H(\Omega)$ *has the Heine-Borel property*. It will then follow from Theorem 1.23 that $H(\Omega)$ *is not locally bounded, hence is not normable.*

Let E be a closed and bounded subset of $H(\Omega)$. Then E satisfies inequalities such as (4) of Section 1.44. Montel's classical theorem about normal families (Th. 14.6 of [23][1]) implies therefore that every sequence $\{f_i\} \subset E$ has a subsequence that converges uniformly on compact subsets of Ω [hence in the topology of $H(\Omega)$] to some $f \in H(\Omega)$. Since E is closed, $f \in E$. This proves that E is compact.

1.46 The spaces $C^\infty(\Omega)$ and $\mathscr{D}_K$ We begin this section by introducing some terminology that will be used in our later work with distributions.

In any discussion of functions of n variables, the term *multi-index* denotes an ordered n-tuple

$$(1) \qquad \alpha = (\alpha_1, \ldots, \alpha_n)$$

of nonnegative integers α_i. With each multi-index α is associated the differential operator

$$(2) \qquad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

whose *order* is

$$(3) \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

If $|\alpha| = 0$, $D^\alpha f = f$.

A complex function f defined in some nonempty open set $\Omega \subset R^n$ is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for every multi-index α.

If K is a compact set in R^n, then $\mathscr{D}_K$ denotes the space of all $f \in C^\infty(R^n)$ whose support lies in K. (The letter $\mathscr{D}$ has been used for these spaces ever since Schwartz published his work on distributions.) If $K \subset \Omega$, then $\mathscr{D}_K$ may be identified with a subspace of $C^\infty(\Omega)$.

We now define a topology on $C^\infty(\Omega)$ which makes $C^\infty(\Omega)$ into a Fréchet space with the Heine-Borel property, such that $\mathscr{D}_K$ is a closed subspace of $C^\infty(\Omega)$ whenever $K \subset \Omega$.

To do this, choose compact sets K_i ($i = 1, 2, 3, \ldots$) such that K_i lies in the interior of K_{i+1} and $\Omega = \bigcup K_i$. Define seminorms p_N on $C^\infty(\Omega)$, $N = 1, 2, 3, \ldots$, by setting

$$(4) \qquad p_N(f) = \max \{ |D^\alpha f(x)| : x \in K_N, |\alpha| \le N \}.$$

They define a metrizable locally convex topology on $C^\infty(\Omega)$; see Theorem 1.37 and remark (*c*) of Section 1.38. For each $x \in \Omega$, the functional $f \to f(x)$ is continuous in this topology. Since $\mathscr{D}_K$ is the intersection of the null spaces of these functionals, as x ranges over the complement of K, it follows that $\mathscr{D}_K$ is closed in $C^\infty(\Omega)$.

A local base is given by the sets

$$(5) \qquad V_N = \left\{ f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N} \right\} \qquad (N = 1, 2, 3, \ldots).$$

If $\{f_i\}$ is a Cauchy sequence in $C^\infty(\Omega)$ (see Section 1.25) and if N is fixed, then $f_i - f_j \in V_N$ if i and j are sufficiently large. Thus $|D^\alpha f_i - D^\alpha f_j| < 1/N$ on K_N, if $|\alpha| \le N$. It follows that each $D^\alpha f_i$ converges (uniformly on compact subsets of Ω) to a function g_α. In particular, $f_i(x) \to g_0(x)$. It is now evident that $g_0 \in C^\infty(\Omega)$, that $g_\alpha = D^\alpha g_0$, and that $f_i \to g$ in the topology of $C^\infty(\Omega)$.

Thus $C^\infty(\Omega)$ is a Fréchet space. The same is true of each of its closed subspaces $\mathscr{D}_K$.

Suppose next that $E \subset C^\infty(\Omega)$ is closed and bounded. By Theorem 1.37, the boundedness of E is equivalent to the existence of numbers $M_N < \infty$ such that $p_N(f) \le M_N$ for $N = 1, 2, 3, \ldots$ and for all $f \in E$. The inequalities $|D^\alpha f| \le M_N$, valid on K_N when $|\alpha| \le N$, imply the equicontinuity of $\{D^\beta f : f \in E\}$ on K_{N-1}, if $|\beta| \le N - 1$. It now follows from Ascoli's theorem (proved in Appendix A) and Cantor's diagonal process

that every sequence in E contains a subsequence $\{f_i\}$ for which $\{D^\beta f_i\}$ converges, uniformly on compact subsets of Ω, for each multi-index β. Hence $\{f_i\}$ converges in the topology of $C^\infty(\Omega)$. This proves that E is compact.

Hence $C^\infty(\Omega)$ has the Heine-Borel property. It follows from Theorem 1.23 that $C^\infty(\Omega)$ *is not locally bounded, hence not normable.* The same conclusion holds for $\mathscr{D}_K$ whenever K has nonempty interior (otherwise $\mathscr{D}_K = \{0\}$), because $\dim \mathscr{D}_K = \infty$ in that case. This last statement is a consequence of the following proposition:

If B_1 and B_2 are concentric closed balls in R^n, with B_1 in the interior of B_2, then there exists $\phi \in C^\infty(R^n)$ such that $\phi(x) = 1$ for every $x \in B_1$, $\phi(x) = 0$ for every x outside B_2, and $0 \leq \phi \leq 1$ on R^n.

To find such a ϕ, we construct $g \in C^\infty(R^1)$ such that $g(x) = 0$ for $x < a$, $g(x) = 1$ for $x > b$ (where $0 < a < b < \infty$ are preassigned) and put

$$\phi(x_1, \ldots, x_n) = 1 - g(x_1^2 + \cdots + x_n^2). \tag{6}$$

The following construction of g has the advantage that suitable choices of $\{\delta_i\}$ can lead to functions with other desired properties.

Suppose $0 < a < b < \infty$. Choose positive numbers $\delta_0, \delta_1, \delta_2, \ldots$, with $\Sigma\delta_i = b - a$; put

$$m_n = \frac{2^n}{\delta_1 \cdots \delta_n} \qquad (n = 1, 2, 3, \ldots); \tag{7}$$

let f_0 be a continuous monotonic function such that $f_0(x) = 0$ when $x < a$, $f_0(x) = 1$ when $x > a + \delta_0$; and define

$$f_n(x) = \frac{1}{\delta_n} \int_{x-\delta_n}^{x} f_{n-1}(t)\, dt \qquad (n = 1, 2, 3, \ldots). \tag{8}$$

Differentiation of this integral shows, by induction, that f_n has n continuous derivatives and that $|D^n f_n| \leq m_n$. If $n > r$, then

$$D^r f_n(x) = \frac{1}{\delta_n} \int_0^{\delta_n} (D^r f_{n-1})(x - t)\, dt, \tag{9}$$

so that

$$|D^r f_n| \leq m_r \qquad (n \geq r), \tag{10}$$

again by induction on n. The mean value theorem, applied to (9), shows that

$$|D^r f_n - D^r f_{n-1}| \leq m_{r+1}\,\delta_n \qquad (n \geq r + 2). \tag{11}$$

Since $\Sigma\delta_n < \infty$, each $\{D^r f_n\}$ converges, uniformly on $(-\infty, \infty)$, as $n \to \infty$. Hence $\{f_n\}$ converges to a function g, with $|D^r g| \leq m_r$ for $r = 1, 2, 3, \ldots$, such that $g(x) = 0$ for $x < a$ and $g(x) = 1$ for $x > b$.

1.47 The spaces L^p with $0 < p < 1$ Consider a fixed p in this range. The elements of L^p are those Lebesgue measurable functions f on $[0, 1]$ for which

$$\Delta(f) = \int_0^1 |f(t)|^p \, dt < \infty, \tag{1}$$

with the usual identification of functions that coincide almost everywhere. Since $0 < p < 1$, the inequality

$$(a + b)^p \leq a^p + b^p \tag{2}$$

holds when $a \geq 0$ and $b \geq 0$. This gives

$$\Delta(f + g) \leq \Delta(f) + \Delta(g), \tag{3}$$

so that

$$d(f, g) = \Delta(f - g) \tag{4}$$

defines an *invariant metric* on L^p. That this d is *complete* is proved in the same way as in the familiar case $p \geq 1$. The balls

$$B_r = \{f \in L^p : \Delta(f) < r\} \tag{5}$$

form a local base for the topology of L^p. Since $B_1 = r^{-1/p} B_r$, for all $r > 0$, B_1 is bounded.

Thus L^p is a locally bounded F-space.

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