**3.1 Definitions** The *dual space* of a topological vector space $X$ is the vector space $X^*$ whose elements are the *continuous* linear functionals on $X$.

Note that addition and scalar multiplication are defined in $X^*$ by

$$(\Lambda_1 + \Lambda_2)x = \Lambda_1 x + \Lambda_2 x, \qquad (\alpha\Lambda)x = \alpha \cdot \Lambda x.$$

It is clear that these operations do indeed make $X^*$ into a vector space.

It will be necessary to use the obvious fact that every complex vector space is also a real vector space, and it will be convenient to use the following (temporary) terminology: An additive functional $\Lambda$ on a complex vector space $X$ is called *real-linear* (*complex-linear*) if $\Lambda(\alpha x) = \alpha\Lambda x$ for every $x \in X$ and for every real (complex) scalar $\alpha$. Our standing rule that any statement about vector spaces in which no scalar field is mentioned applies to both cases is unaffected by this temporary terminology and is still in force.

If $u$ is the real part of a complex-linear functional $f$ on $X$, then $u$ is real-linear and

$$f(x) = u(x) - iu(ix) \qquad (x \in X) \tag{1}$$

because $z = \operatorname{Re} z - i \operatorname{Re}(iz)$ for every $z \in \mathbb{C}$.

Conversely, if $u: X \to R$ is real-linear on a complex vector space $X$ and if $f$ is defined by (1), a straightforward computation shows that $f$ is complex-linear.

Suppose now that $X$ is a complex topological vector space. The above facts imply that a complex-linear functional on $X$ is in $X^*$ if and only if its real part is continuous, and that every continuous real-linear $u: X \to R$ is the real part of a unique $f \in X^*$.

**3.2 Theorem** *Suppose*

(*a*) *$M$ is a subspace of a real vector space $X$,*

(*b*) *$p: X \to R$ satisfies*

$$p(x + y) \le p(x) + p(y) \qquad \textit{and} \qquad p(tx) = p(x)$$

*if $x \in X$, $y \in X$, $t \ge 0$,*

(c) $f: M \to R$ *is linear and* $f(x) \leq p(x)$ *on* $M$.

*Then there exists a linear* $\Lambda: X \to R$ *such that*

$$\Lambda x = f(x) \qquad (x \in M)$$

*and*

$$-p(-x) \leq \Lambda x \leq p(x) \qquad (x \in X).$$

**3.3 Theorem** *Suppose $M$ is a subspace of a vector space $X$, $p$ is a seminorm on $X$, and $f$ is a linear functional on $M$ such that*

$$|f(x)| \leq p(x) \qquad (x \in M).$$

*Then $f$ extends to a linear functional $\Lambda$ on $X$ that satisfies*

$$|\Lambda x| \leq p(x) \qquad (x \in X).$$

PROOF. If the scalar field is $R$, this is contained in Theorem 3.2, since $p$ now satisfies $p(-x) = p(x)$.

Assume that the scalar field is $\mathcal{C}$. Put $u = \text{Re } f$. By Theorem 3.2 there is a real-linear $U$ on $X$ such that $U = u$ on $M$ and $U \leq p$ on $X$. Let $\Lambda$ be the complex-linear functional on $X$ whose real part is $U$. The discussion in Section 3.1 implies that $\Lambda = f$ on $M$.

Finally, to every $x \in X$ corresponds an $\alpha \in \mathcal{C}$, $|\alpha| = 1$, such that $\alpha \Lambda x = |\Lambda x|$. Hence

$$|\Lambda x| = \Lambda(\alpha x) = U(\alpha x) \leq p(\alpha x) = p(x). \qquad ////$$

**Corollary.** *If $X$ is a normed space and $x_0 \in X$, there exists $\Lambda \in X^*$ such that*

$$\Lambda x_0 = \|x_0\| \qquad \textit{and} \qquad |\Lambda x| \leq \|x\| \qquad \textit{for all } x \in X.$$

PROOF. If $x_0 = 0$, take $\Lambda = 0$. If $x_0 \neq 0$, apply Theorem 3.3, with $p(x) = \|x\|$, $M$ the one-dimensional space generated by $x_0$, and $f(\alpha x_0) = \alpha \|x_0\|$ on $M$. ////

**3.4 Theorem** *Suppose $A$ and $B$ are disjoint, nonempty, convex sets in a topological vector space $X$.*

(*a*) *If $A$ is open there exist $\Lambda \in X^*$ and $\gamma \in R$ such that*

$$\mathrm{Re}\ \Lambda x < \gamma \leq \mathrm{Re}\ \Lambda y$$

*for every $x \in A$ and for every $y \in B$.*

(*b*) *If $A$ is compact, $B$ is closed, and $X$ is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1 \in R$, $\gamma_2 \in R$, such that*

$$\mathrm{Re}\ \Lambda x < \gamma_1 < \gamma_2 < \mathrm{Re}\ \Lambda y$$

*for every $x \in A$ and for every $y \in B$.*

Note that this is stated without specifying the scalar field; if it is $R$, then $\mathrm{Re}\ \Lambda = \Lambda$, of course.

**Corollary.** *If $X$ is a locally convex space then $X^*$ separates points on $X$.*

PROOF. If $x_1 \in X$, $x_2 \in X$, and $x_1 \neq x_2$, apply (*b*) of Theorem 3.4 with $A = \{x_1\}$, $B = \{x_2\}$. ////

**3.5 Theorem** *Suppose $M$ is a subspace of a locally convex space $X$, and $x_0 \in X$. If $x_0$ is not in the closure of $M$, then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ but $\Lambda x = 0$ for every $x \in M$.*

PROOF. By (*b*) of Theorem 3.4, with $A = \{x_0\}$ and $B = \bar{M}$, there exists $\Lambda \in X^*$ such that $\Lambda x_0$ and $\Lambda(M)$ are disjoint. Thus $\Lambda(M)$ is a *proper* subspace of the scalar field. This forces $\Lambda(M) = \{0\}$ and $\Lambda x_0 \neq 0$. The desired functional is obtained by dividing $\Lambda$ by $\Lambda x_0$. ////

**Remark.** This theorem is the basis of a standard method of treating certain approximation problems: In order to prove that an $x_0 \in X$ lies in the closure of some subspace $M$ of $X$ it suffices (if $X$ is locally convex) to show that $\Lambda x_0 = 0$ for every continuous linear functional $\Lambda$ on $X$ that vanishes on $M$.

**3.6 Theorem** *If $f$ is a continuous linear functional on a subspace $M$ of a locally convex space $X$, then there exists $\Lambda \in X^*$ such that $\Lambda = f$ on $M$.*

**Remark.** For normed spaces this is an immediate corollary of Theorem 3.3. The general case could also be obtained from 3.3, by relating the continuity of linear functionals to seminorms (see Exercise 8, Chapter 1). The proof given below shows that Theorem 3.6 depends only on the separation property of Theorem 3.5.

PROOF. Assume, without loss of generality, that $f$ is not identically 0 on $M$. Put

$$M_0 = \{x \in M : f(x) = 0\}$$

and pick $x_0 \in M$ such that $f(x_0) = 1$. Since $f$ is continuous, $x_0$ is not in the $M$-closure of $M_0$, and since $M$ inherits its topology from $X$, it follows that $x_0$ is not in the $X$-closure of $M_0$.

Theorem 3.5 therefore assures the existence of a $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ and $\Lambda = 0$ on $M_0$.

If $x \in M$, then $x - f(x)x_0 \in M_0$, since $f(x_0) = 1$. Hence

$$\Lambda x - f(x) = \Lambda x - f(x)\Lambda x_0 = \Lambda(x - f(x)x_0) = 0.$$

Thus $\Lambda = f$ on $M$. ////

We conclude this discussion with another useful corollary of the separation theorem.

**3.7 Theorem** *Suppose $B$ is a convex, balanced, closed set in a locally convex space $X$, $x_0 \in X$, but $x_0 \notin B$. Then there exists $\Lambda \in X^*$ such that $|\Lambda x| \leq 1$ for all $x \in B$, but $\Lambda x_0 > 1$.*

PROOF. Since $B$ is closed and convex, we can apply (*b*) of Theorem 3.4, with $A = \{x_0\}$, to obtain $\Lambda_1 \in X^*$ such that $\Lambda_1 x_0 = re^{i\theta}$ lies outside

the closure $K$ of $\Lambda_1(B)$. Since $B$ is balanced, so is $K$. Hence there exists $s$, $0 < s < r$, so that $|z| \leq s$ for all $z \in K$. The functional $\Lambda = s^{-1}e^{-i\theta}\Lambda_1$ has the desired properties. ////

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