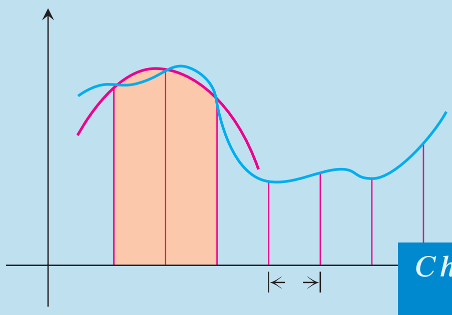


Chapter

8

# TECHNIQUES OF INTEGRATION

**OVERVIEW** The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x)\,dx$$

is equivalent to finding a function $F$ such that $F'(x) = f(x)$, and then adding an arbitrary constant $C$:

$$\int f(x)\,dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

## 8.1 Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$. Success in integration often hinges on the ability to spot what part of the integrand should be called $u$ in order that one will also have $du$, so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

**TABLE 8.1** Basic integration formulas

| | |
|---|---|
| 1. $\int du = u + C$ | 13. $\int \cot u\, du = \ln\lvert\sin u\rvert + C$ <br> $= -\ln\lvert\csc u\rvert + C$ |
| 2. $\int k\, du = ku + C$ (any number $k$) | 14. $\int e^u\, du = e^u + C$ |
| 3. $\int (du + dv) = \int du + \int dv$ | 15. $\int a^u\, du = \frac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$ |
| 4. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$ | 16. $\int \sinh u\, du = \cosh u + C$ |
| 5. $\int \frac{du}{u} = \ln\lvert u\rvert + C$ | 17. $\int \cosh u\, du = \sinh u + C$ |
| 6. $\int \sin u\, du = -\cos u + C$ | 18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ |
| 7. $\int \cos u\, du = \sin u + C$ | 19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ |
| 8. $\int \sec^2 u\, du = \tan u + C$ | 20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{u}{a}\right\rvert + C$ |
| 9. $\int \csc^2 u\, du = -\cot u + C$ | 21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$ $(a > 0)$ |
| 10. $\int \sec u \tan u\, du = \sec u + C$ | 22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$ $(u > a > 0)$ |
| 11. $\int \csc u \cot u\, du = -\csc u + C$ | |
| 12. $\int \tan u\, du = -\ln\lvert\cos u\rvert + C$ <br> $= \ln\lvert\sec u\rvert + C$ | |

We often have to rewrite an integral to match it to a standard formula.

### EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx.$$

**Solution**

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx &= \int \frac{du}{\sqrt{u}} && \begin{array}{l} u = x^2 - 9x + 1, \\ du = (2x-9)\,dx. \end{array} \\
&= \int u^{-1/2}\,du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C && \begin{array}{l} \text{Table 8.1 Formula 4,} \\ \text{with } n = -1/2 \end{array} \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

■

## EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}.$$

**Solution** We complete the square to simplify the denominator:

$$\begin{aligned}
8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\
&= -(x^2 - 8x + 16) + 16 = 16 - (x-4)^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{8x-x^2}} &= \int \frac{dx}{\sqrt{16-(x-4)^2}} \\
&= \int \frac{du}{\sqrt{a^2-u^2}} && \begin{array}{l} a = 4,\ u = (x-4), \\ du = dx \end{array} \\
&= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\
&= \sin^{-1}\left(\frac{x-4}{4}\right) + C.
\end{aligned}$$

■

## EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2\,dx.$$

**Solution** We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\, dx \\ &= 2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx \\ &= 2\tan x + 2\sec x - x + C.\end{aligned}$$

■

## EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

**Solution** We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\ &= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx && \sqrt{u^2} = |u| \\ &= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\ &= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.\end{aligned}$$

■

## EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

**Solution** The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\begin{array}{r} x - 3 \\ 3x + 2\,\overline{)\,3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\,6 \end{array}$$

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

■

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

### EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

**Solution** We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$3\int \frac{x\, dx}{\sqrt{1 - x^2}} = 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du$$

$$= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as $C$ gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.$$ ■

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

HISTORICAL BIOGRAPHY

George David Birkhoff (1884–1944)

### EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x\, dx.$$

**Solution**

$$\int \sec x\, dx = \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{du}{u} \qquad u = \tan x + \sec x, \quad du = (\sec^2 x + \sec x \tan x)\, dx$$

$$= \ln |u| + C = \ln |\sec x + \tan x| + C.$$ ■

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

**TABLE 8.2** The secant and cosecant integrals

1. $$\int \sec u\, du = \ln|\sec u + \tan u| + C$$
2. $$\int \csc u\, du = -\ln|\csc u + \cot u| + C$$

### Procedures for Matching Integrals to Basic Formulas

| PROCEDURE | EXAMPLE |
|---|---|
| Making a simplifying substitution | $\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$ |
| Completing the square | $\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$ |
| Using a trigonometric identity | $(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ <br> $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ <br> $= 2\sec^2 x + 2\sec x\tan x - 1$ |
| Eliminating a square root | $\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$ |
| Reducing an improper fraction | $\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$ |
| Separating a fraction | $\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$ |
| Multiplying by a form of 1 | $\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ <br> $= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$ |

# EXERCISES 8.1

## Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$

10. $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$

14. $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$

16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\displaystyle\int_{\pi/2}^{\pi}(\sin y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$

20. $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$

22. $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$

26. $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6}\frac{dx}{\sqrt{1-9x^2}}$

28. $\displaystyle\int_0^{1}\frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

34. $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}}\frac{dx}{x\cos(\ln x)}$

36. $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

## Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\displaystyle\int_1^{2}\frac{8\,dx}{x^2-2x+2}$

38. $\displaystyle\int_2^{4}\frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

## Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\displaystyle\int (\sec x+\cot x)^2\,dx$

44. $\displaystyle\int (\csc x-\tan x)^2\,dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

## Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\displaystyle\int \frac{x}{x+1}\,dx$

48. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

49. $\displaystyle\int_{\sqrt{2}}^{3}\frac{2x^3}{x^2-1}\,dx$

50. $\displaystyle\int_{-1}^{3}\frac{4x^2-7}{2x+3}\,dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

## Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\displaystyle\int_0^{\pi/4}\frac{1+\sin x}{\cos^2 x}\,dx$

56. $\displaystyle\int_0^{1/2}\frac{2-8x}{1+4x^2}\,dx$

## Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\displaystyle\int \frac{1}{1+\sin x}\,dx$

58. $\displaystyle\int \frac{1}{1+\cos x}\,dx$

59. $\displaystyle\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\displaystyle\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\displaystyle\int \frac{1}{1-\sec x}\,dx$

62. $\displaystyle\int \frac{1}{1-\csc x}\,dx$

## Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\displaystyle\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\displaystyle\int_0^{\pi}\sqrt{1-\cos 2x}\,dx$

**65.** $\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$ **66.** $\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

**67.** $\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$ **68.** $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

**69.** $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$ **70.** $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

## Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

**71.** $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$ **72.** $\int_{0}^{\pi/4} (\sec x + 4\cos x)^2\, dx$

**73.** $\int \cos\theta \csc(\sin\theta)\, d\theta$ **74.** $\int \left(1 + \frac{1}{x}\right) \cot(x + \ln x)\, dx$

**75.** $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

**76.** $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$

**77.** $\int \frac{6\, dy}{\sqrt{y}(1 + y)}$ **78.** $\int \frac{dx}{x\sqrt{4x^2 - 1}}$

**79.** $\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$ **80.** $\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

**81.** $\int \sec^2 t \tan(\tan t)\, dt$ **82.** $\int \frac{dx}{x\sqrt{3 + x^2}}$

## Trigonometric Powers

**83.** **a.** Evaluate $\int \cos^3 \theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)

**b.** Evaluate $\int \cos^5 \theta\, d\theta$.

**c.** Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta\, d\theta$.

**84.** **a.** Evaluate $\int \sin^3 \theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)

**b.** Evaluate $\int \sin^5 \theta\, d\theta$.

**c.** Evaluate $\int \sin^7 \theta\, d\theta$.

**d.** Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13} \theta\, d\theta$.

**85.** **a.** Express $\int \tan^3 \theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)

**b.** Express $\int \tan^5 \theta\, d\theta$ in terms of $\int \tan^3 \theta\, d\theta$.

**c.** Express $\int \tan^7 \theta\, d\theta$ in terms of $\int \tan^5 \theta\, d\theta$.

**d.** Express $\int \tan^{2k+1} \theta\, d\theta$, where $k$ is a positive integer, in terms of $\int \tan^{2k-1} \theta\, d\theta$.

**86.** **a.** Express $\int \cot^3 \theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)

**b.** Express $\int \cot^5 \theta\, d\theta$ in terms of $\int \cot^3 \theta\, d\theta$.

**c.** Express $\int \cot^7 \theta\, d\theta$ in terms of $\int \cot^5 \theta\, d\theta$.

**d.** Express $\int \cot^{2k+1} \theta\, d\theta$, where $k$ is a positive integer, in terms of $\int \cot^{2k-1} \theta\, d\theta$.

## Theory and Examples

**87. Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

**88. Area** Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

**89. Volume** Find the volume of the solid generated by revolving the region in Exercise 87 about the $x$-axis.

**90. Volume** Find the volume of the solid generated by revolving the region in Exercise 88 about the $x$-axis.

**91. Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

**92. Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

**93. Centroid** Find the centroid of the region bounded by the $x$-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

**94. Centroid** Find the centroid of the region that is bounded by the $x$-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

**95. The integral of csc $x$** Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

**96. Using different substitutions** Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

**a.** $u = 1/(x + 1)$

**b.** $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and $-1$

**c.** $u = \tan^{-1} x$

**d.** $u = \tan^{-1}\sqrt{x}$ **e.** $u = \tan^{-1}((x - 1)/2)$

**f.** $u = \cos^{-1} x$ **g.** $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

# 8.2 Integration by Parts

Since

$$\int x\,dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when $f$ can be differentiated repeatedly and $g$ can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

## Product Rule in Integral Form

If $f$ and $g$ are differentiable functions of $x$, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

**Integration by Parts Formula**

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of $u$ and $v$, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for $u$ and $dv$. The next examples illustrate the technique.

### EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

**Solution** We use the formula $\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= \cos x\,dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$

Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Let us examine the choices available for $u$ and $dv$ in Example 1.

### EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x\, dx$.
2. Let $u = x$ and $dv = \cos x\, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x\, dx$ to get $v$.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x)\, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v\, du = \int (x \cos x - x^2 \sin x)\, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x\, dx, \\ du &= -\sin x\, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v\, du = -\int \frac{x^2}{2} \sin x\, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u\, dv$ that we don't see how to evaluate to an integral $\int v\, du$ that we can evaluate. Generally, you choose $dv$ first to be as much of the integrand, including $dx$, as you can readily integrate; $u$ is the leftover part. Keep in mind that integration by parts does not always work.

## EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x\, dx.$$

**Solution** Since $\int \ln x\, dx$ can be written as $\int \ln x \cdot 1\, dx$, we use the formula $\int u\, dv = uv - \int v\, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \dfrac{1}{x}\, dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

**EXAMPLE 4** Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x\,dx.$$

**Solution** With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on $x$ is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\ &= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x\,dx$ in which $n$ is a positive integer, because differentiating $x^n$ will eventually lead to zero and integrating $e^x$ is easy. We say more about this later in this section when we discuss *tabular integration.*

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

**EXAMPLE 5** Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x\,dx.$$

**Solution** Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

■

## Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both $f'$ and $g'$ are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

**Integration by Parts Formula for Definite Integrals**

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

In applying Equation (3), we normally use the $u$ and $v$ notation from Equation (2) because it is easier to remember. Here is an example.

**EXAMPLE 6** Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the $x$-axis from $x = 0$ to $x = 4$.

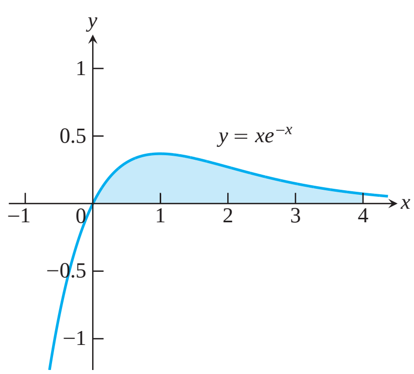



**FIGURE 8.1** The region in Example 6.

**Solution** The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}
\int_0^4 xe^{-x}\, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \\
&= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\, dx \\
&= -4e^{-4} - e^{-x}\Big]_0^4 \\
&= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.
\end{aligned}$$

■

## Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\, dx$, in which $f$ can be differentiated repeatedly to become zero and $g$ can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

### EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

**Solution** With $f(x) = x^2$ and $g(x) = e^x$, we list:

| $f(x)$ and its derivatives | | $g(x)$ and its integrals |
|---|---|---|
| $x^2$ | $(+)$ | $e^x$ |
| $2x$ | $(-)$ | $e^x$ |
| $2$ | $(+)$ | $e^x$ |
| $0$ | | $e^x$ |

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 4. ■

### EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

**Solution** With $f(x) = x^3$ and $g(x) = \sin x$, we list:

| $f(x)$ and its derivatives | | $g(x)$ and its integrals |
|---|---|---|
| $x^3$ | $(+)$ | $\sin x$ |
| $3x^2$ | $(-)$ | $-\cos x$ |
| $6x$ | $(+)$ | $-\sin x$ |
| $6$ | $(-)$ | $\cos x$ |
| $0$ | | $\sin x$ |

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$ ■

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function $f$ nor $g$ can be differentiated repeatedly to become zero.

## Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x)\,dx.$$

(Remember that $g$ may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u\,dv$$

by choosing $dv$ to be part of the integrand including $dx$ and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate $dv$ to get $v$ in order to obtain the right side of the formula

$$\int u\,dv = uv - \int v\,du.$$

If the new integral on the right side is more complex than the original one, try a different choice for $u$ and $dv$.

### EXAMPLE 9 A Reduction Formula

Obtain a "reduction" formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of $\cos x$.

**Solution** We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

Hence

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx, \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by $n$, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When $n$ is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x\,dx = \sin x + C \qquad \text{or} \qquad \int \cos^0 x\,dx = \int dx = x + C.$$

■

**EXAMPLE 10** Using a Reduction Formula

Evaluate

$$\int \cos^3 x\,dx.$$

**Solution** From the result in Example 9,

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

■

# EXERCISES 8.2

## Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta\, d\theta$

3. $\int t^2 \cos t\, dt$

4. $\int x^2 \sin x\, dx$

5. $\int_1^2 x \ln x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

7. $\int \tan^{-1} y\, dy$

8. $\int \sin^{-1} y\, dy$

9. $\int x \sec^2 x\, dx$

10. $\int 4x \sec^2 2x\, dx$

11. $\int x^3 e^x\, dx$

12. $\int p^4 e^{-p}\, dp$

13. $\int (x^2 - 5x)e^x\, dx$

14. $\int (r^2 + r + 1)e^r\, dr$

15. $\int x^5 e^x\, dx$

16. $\int t^2 e^{4t}\, dt$

17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$

18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$

19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$

20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$

21. $\int e^{\theta} \sin \theta\, d\theta$

22. $\int e^{-y} \cos y\, dy$

23. $\int e^{2x} \cos 3x\, dx$

24. $\int e^{-2x} \sin 2x\, dx$

## Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

**25.** $\int e^{\sqrt{3s+9}}\,ds$

**26.** $\int_0^1 x\sqrt{1-x}\,dx$

**27.** $\int_0^{\pi/3} x\tan^2 x\,dx$

**28.** $\int \ln(x+x^2)\,dx$

**29.** $\int \sin(\ln x)\,dx$

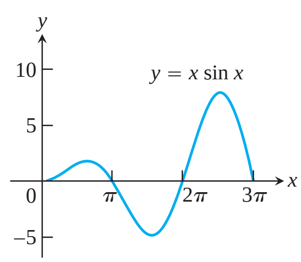
**30.** $\int z(\ln z)^2\,dz$

## Theory and Examples

**31. Finding area** Find the area of the region enclosed by the curve $y = x\sin x$ and the $x$-axis (see the accompanying figure) for

**a.** $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.

**d.** What pattern do you see here? What is the area between the curve and the $x$-axis for $n\pi \le x \le (n+1)\pi$, $n$ an arbitrary nonnegative integer? Give reasons for your answer.




**32. Finding area** Find the area of the region enclosed by the curve $y = x\cos x$ and the $x$-axis (see the accompanying figure) for

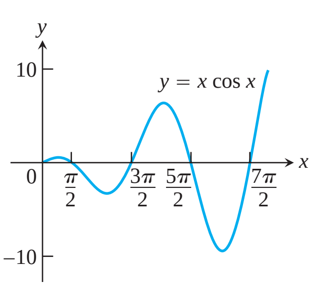
**a.** $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$

**c.** $5\pi/2 \le x \le 7\pi/2$.

**d.** What pattern do you see? What is the area between the curve and the $x$-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

$n$ an arbitrary positive integer? Give reasons for your answer.




**33. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

**34. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

**a.** about the $y$-axis. **b.** about the line $x = 1$.

**35. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

**a.** the $y$-axis. **b.** the line $x = \pi/2$.

**36. Finding volume** Find the volume of the solid generated by revolving the region bounded by the $x$-axis and the curve $y = x\sin x$, $0 \le x \le \pi$, about

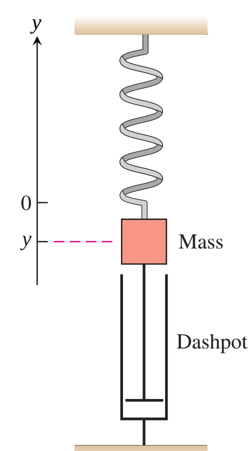
**a.** the $y$-axis. **b.** the line $x = \pi$.

(See Exercise 31 for a graph.)

**37. Average value** A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time $t$ is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

Find the average value of $y$ over the interval $0 \le t \le 2\pi$.




**38. Average value** In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time $t$ is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of $y$ over the interval $0 \le t \le 2\pi$.

## Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

**39.** $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

**40.** $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

**41.** $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

**42.** $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

## Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int yf'(y)\,dy && y = f^{-1}(x), \quad x = f(y),\ dx = f'(y)\,dy\\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with } u = y,\ dv = f'(y)\,dy\\ &= xf^{-1}(x) - \int f(y)\,dy\end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned}\int \ln x\,dx &= \int ye^y\,dy && y = \ln x, \quad x = e^y,\ dx = e^y\,dy\\ &= ye^y - e^y + C\\ &= x\ln x - x + C.\end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned}\int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x\\ &= x\cos^{-1} x - \sin y + C\\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C.\end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of $x$.

**43.** $\int \sin^{-1} x\,dx$

**44.** $\int \tan^{-1} x\,dx$

**45.** $\int \sec^{-1} x\,dx$

**46.** $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when $f^{-1}$ is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of $f^{-1}$ as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

**47.** Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

**a.** $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

**b.** $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

**48.** Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

**a.** $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

**b.** $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to $x$.

**49.** $\int \sinh^{-1} x\,dx$

**50.** $\int \tanh^{-1} x\,dx$