Solution of Heat Equation

In the previous section we applied separation of variables to solve the heat conduction equation

$$u_t = cu_{xx}\,, \quad 0 < x < L\,, t > 0 \quad c > 0$$

$$u(0,t) = u(L,t) = 0 \text{ and } u(x,0) = f(x)$$

The following examples explain how to do it.

Example 1: Apply the method of separation of variables to solve the heat equation $u_t = 2u_{xx}$ over $0 < x < 3\,, t > 0$ for the boundary conditions $u(0,t) = u(3,t) = 0$ and the initial condition $u(x,0) = 5\sin 4\pi x$

Solution : Assume the solution is $u(x,t) = X(x)T(t)$

$u_t = XT'$ and $u_{xx} = X''T$

Substituting in the given equation, we have

$$XT' = 2X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{2T} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2 \quad \text{and} \quad \frac{T'}{2T} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T' + 2T\lambda^2 = 0$$

$$X = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad T = Ce^{-2\lambda^2 t}$$

So $u(x,t) = (A\sin\lambda x + B\cos\lambda x) \times Ce^{-2\lambda^2 t}$

Or $u(x,t) = e^{-2\lambda^2 t}(D\sin\lambda x + E\cos\lambda x)$; $AC = D\,,$ $BC = E$

$u(0,t) = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $E = 0$

$u(3,t) = 0 \Rightarrow De^{-2\lambda^2 t}\sin 3\lambda = 0 \Rightarrow e^{-2\lambda^2 t} \neq 0$

$D = 0 \Rightarrow u = 0$ (trivial solution)

The only sensible deduction is that $\sin 3\lambda = 0 \Rightarrow 3\lambda = n\pi \Rightarrow \lambda = n\pi/3$

Then $u(x,t) = D_n e^{-\frac{2n^2\pi^2}{9}t}\sin\frac{n\pi}{3}x$

$u(x,0) = 5\sin 4\pi x \Rightarrow 5\sin 4\pi x = D_n \sin\frac{n\pi}{3}x$

$4\pi = \frac{n\pi}{3} \Rightarrow n = 12$ so $D_{12} = 5$

Then $u(x,t) = 5e^{-\frac{288\pi^2}{9}t}\sin(4\pi x) = 5e^{-32\pi^2 t}\sin(4\pi x)$

Example 2: Solve by the method of separation of variables the heat equation

$$u_t = u_{xx} \;;\; 0 < x < 1 \;, t > 0 \text{ with } u_x(0,t) = u_x(1,t) = 0$$

and the initial condition $u(x,0) = 3\cos 2\pi x$

Solution : Assume the solution is $u(x,t) = X(x)T(t)$

$$u_t = XT' \quad \text{and} \quad u_{xx} = X''T$$

$$XT' = X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2 \quad \text{and} \quad \frac{T'}{T} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T' + T\lambda^2 = 0$$

$$X = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad T = Ce^{-\lambda^2 t}$$

So $u(x,t) = (A\sin\lambda x + B\cos\lambda x) \times Ce^{-\lambda^2 t}$

Or $u(x,t) = e^{-\lambda^2 t}(D\sin\lambda x + E\cos\lambda x) \;;\; AC = D, \quad BC = E$

$$u_x(x,t) = e^{-\lambda^2 t}(D\lambda\cos\lambda x - E\lambda\sin\lambda x)$$

$$u_x(0,t) = 0 \quad \Rightarrow D = 0$$

$$u_x(1,t) = 0 \Rightarrow -E\lambda e^{-\lambda^2 t}\sin\lambda = 0 \quad \Rightarrow \quad \lambda e^{-\lambda^2 t} \neq 0$$

$E = 0 \Rightarrow u = 0$ (trivial solution)

$$\sin\lambda = 0 \quad \Rightarrow \quad \lambda = n\pi, \quad n = 1,2,3,\cdots$$

Then $u(x,t) = E_n e^{-n^2\pi^2 t}\cos n\pi x$

$$u(x,0) = 3\cos 2\pi x \Rightarrow \quad 3\cos 2\pi x = E_n\cos n\pi x$$

$2\pi = n\pi \quad \Rightarrow \quad n = 2$ so $E_2 = 3$

Then $u(x,t) = 3e^{-4\pi^2 t}\cos(2\pi x)$

Example 3: Apply the method of separation of variables to solve the heat equation

$u_t = 3u_{xx}$ over $0 < x < \pi , t > 0$ for the boundary conditions

$u(0,t) = u(\pi,t) = 0$ and the initial condition $u(x,0) = 3\sin 2x - 6\sin 5x$

Solution : Assume the solution is $u(x,y) = X(x)Y(y)$ then

$u_t = XT'$ and $u_{xx} = X''T$

$$XT' = 3X''T \Rightarrow \frac{X''}{X} = \frac{T'}{3T} = -\lambda^2$$

$$\frac{X''}{X} = -\lambda^2 \quad \text{and} \quad \frac{T'}{3T} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T' + 3T\lambda^2 = 0$$

$$X = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad T = Ce^{-3\lambda^2 t}$$

So $u(x,t) = (A\sin\lambda x + B\cos\lambda x) \times Ce^{-3\lambda^2 t}$

Or $u(x,t) = e^{-3\lambda^2 t}(D\sin\lambda x + E\cos\lambda x)$; $AC = D$, $BC = E$

$u(0,t) = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $E = 0$

$u(\pi,t) = 0 \Rightarrow De^{-2\lambda^2 t}\sin\pi\lambda = 0$; $e^{-2\lambda^2 t} \neq 0$

$D = 0 \Rightarrow u = 0$ (trivial solution)

$\therefore \sin(\lambda\pi) = 0 \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$, $n = 1,2,3,\cdots$

Then $u(x,t) = D_n e^{-3n^2 t}\sin nx$

$u(x,0) = 3\sin 2x - 6\sin 5x$

$3\sin 2x - 6\sin 5x = D_2\sin 2x + D_5\sin 5x$

$D_2 = 3 , D_5 = -6$

Then $u(x,t) = 3e^{-12t}\sin 2x - 6e^{-75t}\sin 5x$

H.W: Apply the method of separation of variables to solve the heat equation

1. $u_t = u_{xx}$; $0 < x < 1$, $t > 0$ with $u(0,t) = u_x(1,t) = 0$

   and the initial condition $u(x,0) = 5\sin\frac{3\pi}{2}x$

2. $u_t = u_{xx}$; $0 < x < 1$, $t > 0$ with $u_x(0,t) = u(1,t) = 0$

   and the initial condition $u(x,0) = 4\cos\frac{5\pi}{2}x$