Properties of Matrix Arithmetic

I've given examples which illustrate how you can do arithmetic with matrices. Now I'll give *precise* definitions of the various matrix operations. This will allow me to *prove* some useful properties of these operations.

If A is a matrix, the element in the i^{th} row and j^{th} column will be denoted A_{ij}. (Sometimes I'll switch to lower-case letters and use a_{ij} instead of A_{ij}). Thus,

$(A+B)_{ij}$ is the $(i,j)^{\text{th}}$ entry of $A+B$.

$(AB)_{ij}$ is the $(i,j)^{\text{th}}$ entry of AB.

$(A^T)_{ij}$ is the $(i,j)^{\text{th}}$ entry of A^T.

Remark. To avoid confusion, use a comma between the indices where appropriate. "A_{24}" clearly means the entry in row 2, column 4. However, for the entry in row 13, column 6, write "$A_{13,6}$", not "A_{136}".$\lrcorner$

Here are the formal definitions of the matrix operations. When I write something like "for all i and j", you should take this to mean "for all i such that $1 \le i \le m$, where m is the number of rows, and for all j such that $1 \le j \le n$, where n is the number of columns".

Definition. (*Equality*) Let A and B be matrices. Then $A = B$ if and only if A and B have the same dimensions and $a_{ij} = b_{ij}$ for all i and j.

This definition says that two matrices are equal if they have the same dimensions and corresponding entries are equal.

Definition. (*Sums and Differences*) Let A and B be matrices. If A and B have the same dimensions, then the *sum* $A+B$ and the *difference* $A-B$ are defined, and their entries are given by

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (A-B)_{ij} = A_{ij} - B_{ij} \quad \text{for all} \quad i, j.$$

This definition says that if two matrices have the same dimensions, you can add or subtract them by adding or subtracting corresponding entries.

Definition. The $m \times n$ *zero matrix* 0 is the $m \times n$ matrix whose $(i,j)^{\text{th}}$ entry is given by

$$0_{ij} = 0 \quad \text{for all} \quad i, j.$$

Proposition. Let A, B, and C be $m \times n$ matrices, and let 0 denote the $m \times n$ zero matrix. Then:

(a) (*Associativity of Addition*)

$$(A+B)+C = A+(B+C).$$

(b) (*Commutativity of Addition*)

$$A+B = B+A.$$

(c) (*Identity for Addition*)

$$0+A = A \quad \text{and} \quad A+0 = A.$$

Proof. Each of the properties is a matrix equation. The definition of matrix equality says that I can prove that two matrices are equal by proving that their corresponding entries are equal. I'll follow this strategy in each of the proofs that follows.

(a) To prove that $(A+B)+C = A+(B+C)$, I have to show that their corresponding entries are equal:

$$((A+B)+C)_{ij} = (A+(B+C))_{ij}.$$

(Do you understand what this says? $((A+B)+C)_{ij}$ is the $(i,j)^{\text{th}}$ entry of $(A+B)+C$, while $(A+(B+C))_{ij}$ is the $(i,j)^{\text{th}}$ entry of $A+(B+C)$.)

Since this is the first proof of this kind that I've done, I'll show the justification for each step.

$$\begin{aligned} ((A+B)+C)_{ij} &= (A+B)_{ij} + C_{ij} && (\text{Def. of addition}) \\ &= (A_{ij}+B_{ij}) + C_{ij} && (\text{Def. of addition}) \\ &= A_{ij} + (B_{ij}+C_{ij}) && (\text{Associativity}) \\ &= A_{ij} + (B+C)_{ij} && (\text{Def. of addition}) \\ &= (A+(B+C))_{ij} && (\text{Def. of addition}) \end{aligned}$$

"Associativity" refers to associativity of addition for *numbers*.

Therefore, $(A+B)+C = A+(B+C)$, because their corresponding elements are equal.

(b) To prove that $A+B = B+A$, I have to show that their corresponding entries are equal:

$$(A+B)_{ij} = (B+A)_{ij}.$$

By definition of matrix addition,

$$\begin{aligned}(A+B)_{ij} &= A_{ij}+B_{ij} \quad &&(\text{Def. of addition})\\ &= B_{ij}+A_{ij} &&(\text{Commutativity})\\ &= (B+A)_{ij} &&(\text{Def. of addition})\end{aligned}$$

"Commutativity" refers to commutativity of addition of *numbers*.

Therefore,

$$A+B=B+A.$$

(c) To prove that $A+0=A$, I have to show that their corresponding entries are equal:

$$(A+0)_{ij}=A_{ij}.$$

By definition of matrix addition and the zero matrix,

$$\begin{aligned}(A+0)_{ij} &= A_{ij}+0_{ij} \quad &&(\text{Def. of addition})\\ &= A_{ij}+0 &&(\text{Def. of zero matrix})\\ &= A_{ij} &&(\text{Arithmetic})\end{aligned}$$

"Arithmetic" refers to the fact that if x is a number, then $x+0=x$.

Therefore, $A+0=A$.

In part (b), I showed that addition is commutative. Therefore,

$$0+A=A+0=A. \quad \square$$

Remark. You can see that the idea in many of these proofs for *matrices* is to reduce the proof to a known property of *numbers* (such as associativity or commutativity) by looking at the entries of the matrices. Since most of these proofs are fairly simple, I won't write out all of them, and I won't do them in step-by-step detail like the ones above. You should try working through some of them yourself to ensure that you get the idea.□

Definition. (*Multiplication by Numbers*) If A is a matrix and k is a number, then kA is the matrix having the same dimensions as A, and whose entries are given by

$$(kA)_{ij}=k\cdot A_{ij} \quad \text{for all} \quad i,j.$$

(It's considered ugly to write a number on the *right* side of a matrix if you want to multiply. For the record, I'll define Ak to be the same as kA .)

This definition says that to multiply a matrix by a number, multiply each entry by the number.

Definition. If A is a matrix, then $-A$ is the matrix having the same dimensions as A, and whose entries are given by

$$(-A)_{ij} = -A_{ij}.$$

Proposition. Let A and B be matrices with the same dimensions, and let k be a number. Then:

(a) $k(A+B) = kA + kB$ and $k(A-B) = kA - kB$.

(b) $0 \cdot A = 0$.

(c) $1 \cdot A = A$.

(d) $(-1) \cdot A = -A$.

(e) $A - B = A + (-B)$.

Note that in (b), the 0 on the left is the *number* 0, while the 0 on the right is the zero matrix.

Proof. I'll prove (a) by way of example and leave the proofs of the other parts to you.

First, I want to show that $k(A+B) = kA + kB$. I have to show that corresponding entries are equal, i.e.

$$(k(A+B))_{ij} = (kA + kB)_{ij}.$$

I apply the definitions of matrix addition and multiplication of a matrix by a number:

$$(k(A+B))_{ij} = k \cdot (A+B)_{ij} = k(A_{ij} + B_{ij}) = kA_{ij} + kB_{ij},$$

$$(kA + kB)_{ij} = (kA)_{ij} + (kB)_{ij} = kA_{ij} + kB_{ij}.$$

Therefore, $(k(A+B))_{ij} = (kA+kB)_{ij}$, so $k(A+B) = kA + kB$.

Next, I want to show that $k(A-B) = kA - kB$. I just repeat the last proof with "-" in place of "+".

I have to show that corresponding entries are equal, i.e.

$$(k(A-B))_{ij} = (kA - kB)_{ij}.$$

I apply the definitions of matrix subtraction and multiplication of a matrix by a number:

$$(k(A-B))_{ij} = k \cdot (A-B)_{ij} = k(A_{ij} - B_{ij}) = kA_{ij} - kB_{ij},$$

$$(kA - kB)_{ij} = (kA)_{ij} - (kB)_{ij} = kA_{ij} - kB_{ij}.$$

Therefore, $(k(A-B))_{ij} = (kA - kB)_{ij}$, so $k(A-B) = kA - kB$.$\sqcup$

Suppose A and B are matrices with compatible dimensions for multiplication. Where does the $(i,j)^{\text{th}}$ entry of AB come from? It comes from multiplying the i^{th} row of A with the j^{th} column of B:

$$\begin{array}{c} \\ i \end{array} \begin{bmatrix} \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{in} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{array}{c} \quad j \\ \begin{bmatrix} \cdots & B_{1j} & \cdots \\ \cdots & B_{2j} & \cdots \\ & \vdots & \\ \cdots & B_{nj} & \cdots \end{bmatrix} \end{array}$$

Corresponding elements are multiplied, and then the products are summed. In equation form, this means that the $(i,j)^{\text{th}}$ entry of the product is

$$\sum_{k=1}^{n} A_{ik} B_{kj}.$$

Definition. (*Multiplication*) Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. The *product* AB is the $m \times p$ matrix whose $(i,j)^{\text{th}}$ entry is given by

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \quad \text{for all} \quad i, j.$$

It's often useful to have a symbol which you can use to compare two quantities i and j --- specifically, a symbol which equals 1 when $i = j$ and equals 0 when $i \neq j$.

Definition. The $n \times n$ *identity matrix* I_n (or just I, if there's no risk of confusion) is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is given by

$$I_{ij} = \delta_{ij} \quad \text{for all} \quad i, j.$$

Proposition.

(a) (*Associativity of Matrix Multiplication*) If A, B, and C are matrices which are compatible for multiplication, then

$$(AB)C = A(BC).$$

(b) (*Distributivity of Multiplication over Addition*) If A, B, C, D, E, and F are matrices compatible for addition and multiplication, then

$$A(B+C) = AB + AC \quad \text{and} \quad (D+E)F = DF + EF.$$

(c) If j and k are numbers and A and B are matrices which are compatible for multiplication, then

$$k(AB) = (kA)B = A(kB) \quad \text{and} \quad (jk)A = j(kA).$$

(d) (*Identity for Multiplication*) If A is an $m \times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_m A = A.$$

The "compatible for addition" and "compatible for multiplication" assumptions mean that the matrices should have dimensions which make the operations in the equations *legal* --- but otherwise, there are no restrictions on what the dimensions can be.

Proof. I'll prove (a) and part of (d) by way of example, and leave the proofs of the other parts to you.

Before starting, I should say that this proof is rather technical, but try to follow along as best you can. I'll use i, j, k, and l as subscripts.

Suppose that A is an $m \times n$ matrix, B is an $n \times p$ matrix, and C is a $p \times q$ matrix. I want to prove that $(AB)C = A(BC)$. I have to show that corresponding entries are equal, i.e.

$$((AB)C)_{il} = (A(BC))_{il}.$$

By definition of matrix multiplication,

$$((AB)C)_{il} = \sum_{k=1}^{p} (AB)_{ik} C_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl},$$

$$(A(BC))_{il} = \sum_{j=1}^{n} A_{ij} (BC)_{jl} = \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk} C_{kl} \right).$$

If you stare at those two terrible double sums for a while, you can see that they involve the same A, B, and C terms, and they involve the same summations --- but in different orders. I'm allowed to convert one into the other by *interchanging the order of summation*, and using the distributive law:

$$\sum_{k=1}^{p}\left(\sum_{j=1}^{n} A_{ij}B_{jk}\right)C_{kl} = \sum_{k=1}^{p}\sum_{j=1}^{n}(A_{ij}B_{jk}C_{kl}) = \sum_{j=1}^{n}\sum_{k=1}^{p}(A_{ij}B_{jk}C_{kl}) = \sum_{j=1}^{n} A_{ij}\left(\sum_{k=1}^{p} B_{jk}C_{kl}\right).$$

Therefore, $((AB)C)_{il} = (A(BC))_{il}$, and so $(AB)C = A(BC)$. Wow!

Next, I'll prove the second part of (d), namely that $I_m A = A$. As usual, I must show that corresponding entries are equal:

$$(I_m A)_{ij} = A_{ij}.$$

By definition of matrix multiplication and the identity matrix,

$$(I_m A)_{ij} = \sum_{k=1}^{m} \delta_{ik} A_{kj}.$$

Using the lemma I proved on the Kronecker delta, I get

$$\sum_{k=1}^{m} \delta_{ik} A_{kj} = A_{ij}.$$

Thus, $(I_m A)_{ij} = A_{ij}$, and so $I_m A = A$.□

Definition. Let A be an $m \times n$ matrix. The *transpose* of A is the $n \times m$ matrix whose $(i, j)^{\text{th}}$ entry is given by

$$(A^T)_{ij} = A_{ji} \quad \text{for all} \quad i, j.$$

Proposition. Let A and B be matrices of the same dimension, and let k be a number. Then:

(a) $(A^T)^T = A$.

(b) $(A + B)^T = A^T + B^T$.

(c) $(kA)^T = kA^T$.

Proof. I'll prove (b) by way of example and leave the proofs of the other parts for you.

I want to show that $(A + B)^T = A^T + B^T$. I have to show the corresponding entries are equal:

$$\left((A + B)^T\right)_{ij} = (A^T + B^T)_{ij}.$$

Now

$$\left((A+B)^T\right)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji},$$

$$(A^T + B^T)_{ij} = (A^T)_{ij} + (B^T)_{ij} = A_{ji} + B_{ji}.$$

Thus, $\left((A+B)^T\right)_{ij} = (A^T + B^T)_{ij}$, so $(A+B)^T = A^T + B^T$.□

Proposition. Suppose A and B are matrices which are compatible for multiplication. Then

$$(AB)^T = B^T A^T.$$

Proof. I'll derive this using the matrix multiplication formula.

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Let A^T_{ij} denote the $(i,j)^{\text{th}}$ entry of A^T, and likewise for B and AB. Then

$$(AB)^T_{ji} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} A^T_{ki} B^T_{jk} = \sum_{k=1}^{n} B^T_{jk} A^T_{ki}.$$

The product on the right is the $(j,i)^{\text{th}}$ entry of $B^T A^T$, while $(AB)^T_{ji}$ is the $(j,i)^{\text{th}}$ entry of $(AB)^T$. Therefore, $(AB)^T = B^T A^T$, since their corresponding entries are equal.□

Definition.

(a) A matrix X is *symmetric* if $X^T = X$.

(b) A matrix X is *skew symmetric* if $X^T = -X$.

Remarks. Both definitions imply that X is a square matrix.

In terms of elements, X is symmetric if

$$X_{ij} = X_{ji} \quad \text{for all} \quad i, j.$$

X is skew symmetric if

$$X_{ij} = -X_{ji} \quad \text{for all} \quad i, j. \quad □$$

Example. A symmetric matrix is symmetric across its *main diagonal* (the diagonal running from northwest to southeast). For example,

$$\begin{bmatrix} 0 & 2 & -9 \\ 2 & \sqrt{3} & 4 \\ -9 & 4 & 5 \end{bmatrix}$$

is symmetric.

Here's a skew symmetric matrix:

$$\begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & 17 \\ 2 & -17 & 0 \end{bmatrix}.$$

Entries which are symmetrically located across the main diagonals are negatives of one another. The entries on the main diagonal must be 0, since they must be equal to their negatives.□

The next result is pretty easy, but it illustrates how you can use the definitions of symmetry and skew symmetry in writing proofs. Notice that I'm *not* writing out the *entries* of the matrices!

Proposition.

(a) The sum of symmetric matrices is symmetric.

(b) The sum of skew symmetric matrices is skew symmetric.

Proof. (a) Let A and B be symmetric. I must show that $A+B$ is symmetric. Now

$$(A+B)^T = A^T + B^T = A + B.$$

The first equality follows from a property I proved for transposes. The second equality follows from the fact that A is symmetric (so $A^T = A$) and B is symmetric (so $B^T = B$).

Since $(A+B)^T = A+B$, it follows that $A+B$ is symmetric.

(b) Let A and B be skew symmetric, so $A^T = -A$ and $B^T = -B$. I must show that $A+B$ is skew symmetric. Now

$$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B).$$

Therefore, $A+B$ is skew symmetric.□