

# 1 Classical Electromagnetic Fields

In this book we present the basic ideas needed to understand how laser light interacts with various forms of matter. Among the important consequences is an understanding of the laser itself. The present chapter summarizes classical electromagnetic fields, which describe laser light remarkably well. The chapter also discusses the interaction of these fields with a medium consisting of classical simple harmonic oscillators. It is surprising how well this simple model describes linear absorption, a point discussed from a quantum mechanical point of view in Sect. 3.3. The rest of the book is concerned with *nonlinear* interactions of radiation with matter. Chapter 2 generalizes the classical oscillator to treat simple kinds of nonlinear mechanisms, and shows us a number of phenomena in a relatively simple context. Starting with Chap. 3, we treat the medium quantum mechanically. The combination of a classical description of light and a quantum mechanical description of matter is called the *semiclassical* approximation. This approximation is not always justified (Chaps. 13–19), but there are remarkably few cases in quantum optics where we need to quantize the field.

In the present chapter, we limit ourselves both to *classical* electromagnetic fields and to *classical* media. Section 1.1 briefly reviews Maxwell's equations in a vacuum. We derive the wave equation, and introduce the slowly-varying amplitude and phase approximation for the electromagnetic field. Section 1.2 recalls Maxwell's equations in a medium. We then show the roles of the in-phase and in-quadrature parts of the polarization of the medium through which the light propagates, and give a brief discussion of Beer's law of light absorption. Section 1.3 discusses the classical dipole oscillator. We introduce the concept of the self-field and show how it leads to radiative damping. Then we consider the classical Rabi problem, which allows us to introduce the classical analog of the optical Bloch equations. The derivations in Sects. 1.1–1.3 are not necessarily the simplest ones, but they correspond as closely as possible to their quantum mechanical counterparts that appear later in the book.

Section 1.4 is concerned with the coherence of the electromagnetic field. We review the Young and Hanbury Brown-Twiss experiments. We introduce the notion of  $n$ th order coherence. We conclude this section by a brief

comment on antibunching, which provides us with a powerful test of the quantum nature of light.

With knowledge of Sects. 1.1–1.4, we have all the elements needed to understand an elementary treatment of the Free-Electron Laser (FEL), which is presented in Sect. 1.5. The FEL is in some way the simplest laser to understand, since it can largely be described classically, i.e., there is no need to quantize the matter.

## 1.1 Maxwell's Equations in a Vacuum

In the absence of charges and currents, Maxwell's equations are given by

$$\nabla \cdot \mathbf{B} = 0, \quad (1.1)$$

$$\nabla \cdot \mathbf{E} = 0, \quad (1.2)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (1.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (1.4)$$

where  $\mathbf{E}$  is the electric field,  $\mathbf{B}$  is the magnetic field,  $\mu_0$  is the permeability of the free space, and  $\epsilon_0$  is the permittivity of free space (in this book we use MKS units throughout). Alternatively it is useful to write  $c^2$  for  $1/\mu_0\epsilon_0$ , where  $c$  is the speed of light in the vacuum. Taking the curl of (1.3) and substituting the rate of change of (1.4) we find

$$\nabla \times \nabla \times \mathbf{E} = - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (1.5)$$

This equation can be simplified by noting that  $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$  and using (1.2). We find the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \quad (1.6)$$

This tells us how an electromagnetic wave propagates in a vacuum. By direct substitution, we can show that

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 f(\mathbf{K} \cdot \mathbf{r} - vt) \quad (1.7)$$

is a solution of (1.6) where  $f$  is an arbitrary function,  $\mathbf{E}_0$  is a constant,  $v$  is an oscillation frequency in radians/second ( $2\pi \times \text{Hz}$ ),  $\mathbf{K}$  is a constant vector in the direction of propagation of the field, and having the magnitude  $K \equiv |\mathbf{K}| = v/c$ . This solution represents a transverse plane wave propagating along the direction of  $\mathbf{K}$  with speed  $c = v/K$ .

A property of the wave equation (1.6) is that if  $\mathbf{E}_1(\mathbf{r}, t)$  and  $\mathbf{E}_2(\mathbf{r}, t)$  are solutions, then the superposition  $a_1 \mathbf{E}_1(\mathbf{r}, t) + a_2 \mathbf{E}_2(\mathbf{r}, t)$  is also a solution,

where  $a_1$  and  $a_2$  are any two constants. This is called the principle of superposition. It is a direct consequence of the fact that differentiation is a linear operation. In particular, the superposition

$$\mathbf{E}(\mathbf{r}, t) = \sum_k \mathbf{E}_k f(\mathbf{K}_k \cdot \mathbf{r} - vt) \quad (1.8)$$

is also a solution. This shows us that nonplane waves are also solutions of the wave equation (1.6).

Quantum opticians like to decompose electric fields into “positive” and “negative” frequency parts

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^+(\mathbf{r}, t) + \mathbf{E}^-(\mathbf{r}, t), \quad (1.9)$$

where  $\mathbf{E}^+(\mathbf{r}, t)$  has the form

$$\mathbf{E}^+(\mathbf{r}, t) = \frac{1}{2} \sum_n \mathbf{E}_n(\mathbf{r}) e^{-i\nu_n t}, \quad (1.10)$$

where  $E_n(\mathbf{r})$  is a complex function of  $\mathbf{r}$ ,  $\nu_n$  is the corresponding frequency, and in general

$$\mathbf{E}^-(\mathbf{r}, t) = [\mathbf{E}^+(\mathbf{r}, t)]^*. \quad (1.11)$$

In itself this decomposition is just that of the analytic signal used in classical coherence theory [see Born and Wolf (1970)], but as we see in Chap. 13, it has deep foundations in the quantum theory of light detection. For now we consider this to be a convenient mathematical trick that allows us to work with exponentials rather than with sines and cosines. It is easy to see that since the wave equation (1.6) is real, if  $\mathbf{E}^+(\mathbf{r}, t)$  is a solution, then so is  $\mathbf{E}^-(\mathbf{r}, t)$ , and the linearity of (1.6) guarantees that the sum (1.9) is also a solution.

In this book, we are concerned mostly with the interaction of monochromatic (or quasi-monochromatic) laser light with matter. In particular, consider a linearly-polarized plane wave propagating in the  $z$ -direction. Its electric field can be described by

$$\mathbf{E}^+(z, t) = \frac{1}{2} \hat{\mathbf{x}} E_0(z, t) e^{i[Kz - \nu t - \varphi(z, t)]}, \quad (1.12)$$

where  $\hat{\mathbf{x}}$  is the direction of polarization,  $E_0(z, t)$  is a real amplitude,  $\nu$  is the central frequency of the field, and the wave number  $K = \nu/c$ . If  $E(z, t)$  is truly monochromatic,  $E_0$  and  $\varphi$  are constants in time and space. More generally, we suppose they vary sufficiently slowly in time and space that the following inequalities are valid:

$$\frac{\partial E_0}{\partial t} = v E_0, \quad (1.13)$$

$$\frac{\partial E_0}{\partial z} = K E_0, \quad (1.14)$$

$$\frac{\partial \phi}{\partial t} = v, \quad (1.15)$$

$$\frac{\partial \phi}{\partial z} = K. \quad (1.16)$$

These equations define the so-called *slowly-varying amplitude and phase approximation (SVAP)*, which plays a central role in laser physics and pulse propagation problems. Physically it means that we consider light waves whose amplitudes and phases vary little within an optical period and an optical wavelength. Sometimes this approximation is called the SVEA, for *slowly-varying envelope approximation*.

The SVAP leads to major mathematical simplifications as can be seen by substituting the field (1.12) into the wave equation (1.6) and using (1.13–1.16) to eliminate the small contributions  $\ddot{E}_0$ ,  $\ddot{\phi}$ ,  $E_0''$ ,  $\phi''$ , and  $\dot{E}_0 \dot{\phi}$ . We find

$$\frac{\partial E_0}{\partial z} + \frac{1}{c} \frac{\partial E_0}{\partial t} = 0, \quad (1.17)$$

$$\frac{\partial \phi}{\partial z} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0, \quad (1.18)$$

where (1.17) results from equating the sum of the imaginary parts to zero and (1.18) from the real parts. Thus the SVAP allows us to transform the second-order wave equation (1.6) into first-order equations. Although this does not seem like much of an achievement right now, since we can solve (1.6) exactly anyway, it is a tremendous help when we consider Maxwell's equations in a medium. The SVAP is not always a good approximation. For example, plasma physicists who shine light on targets typically must use the second-order equations. In addition, the SVAP approximation also neglects the backward propagation of light.

## 1.2 Maxwell's Equations in a Medium

Inside a macroscopic medium, Maxwell's equations (1.1–1.4) become

$$\nabla \cdot \mathbf{B} = 0, \quad (1.19)$$

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}, \quad (1.20)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (1.21)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.22)$$

These equations are often called the *macroscopic* Maxwell's equations, since they relate vectors that are averaged over volumes containing many atoms but which have linear dimensions small compared to significant variations in the applied electric field. General derivations of (1.19–1.22) can be very complicated, but the discussion by Jackson (1999) is quite readable. In (1.20, 1.22), the displacement electric field  $\mathbf{D}$  is given for our purpose by

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P} , \quad (1.23)$$

where the permittivity  $\epsilon$  includes the contributions of the host lattice and  $\mathbf{P}$  is the induced polarization of the resonant or nearly resonant medium we wish to treat explicitly. For example, in ruby the  $\text{Al}_2\text{O}_3$  lattice has an index of refraction of 1.76, which is included in  $\epsilon$ . The ruby color is given by  $\text{Cr}$  ions which are responsible for laser action. We describe their interaction with light by the polarization  $\mathbf{P}$ . Indeed much of this book deals with the calculation of  $\mathbf{P}$  for various situations. The free charge density  $\rho_{\text{free}}$  in (1.20) consists of all charges other than the bound charges inside atoms and molecules, whose effects are provided for by  $\mathbf{P}$ . We don't need  $\rho_{\text{free}}$  in this book. In (1.22), the magnetic field  $\mathbf{H}$  is given by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} - \mathbf{M} , \quad (1.24)$$

where  $\mu$  is the permeability of the host medium and  $\mathbf{M}$  is the magnetization of the medium. For the media we consider,  $\mathbf{M} = 0$  and  $\mu = \mu_0$ . The current density  $\mathbf{J}$  is often related to the applied electric field  $\mathbf{E}$  by the constitutive relation  $\mathbf{J} = \sigma \mathbf{E}$ , where  $\sigma$  is the conductivity of the medium.

The macroscopic wave equation corresponding to (1.6) is given by combining the curl of (1.21) with (1.23, 1.24). In the process we find  $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla^2 \mathbf{E} - \nabla^2 \mathbf{E} = 0$ . In optics  $\nabla \cdot \mathbf{E} = 0$ , since most light field vectors vary little along the directions in which they point. For example, a plane-wave field is constant along the direction it points, causing its  $\nabla \cdot \mathbf{E}$  to vanish identically. We find

$$-\nabla^2 \mathbf{E} + \mu \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial^2 \mathbf{P}}{\partial t^2} , \quad (1.25)$$

where  $c = 1/\sqrt{\epsilon\mu}$  is now the speed of light in the host medium. In Chap. 7 we use the  $\partial \mathbf{J}/\partial t$  term to simulate losses in a Fabry–Perot resonator. We drop this term in our present discussion.

For a quasi-monochromatic field, the polarization induced in the medium is also quasi-monochromatic, but generally has a different phase from the field. Thus as for the field (1.9) we decompose the polarization into positive and negative frequency parts

$$\mathbf{P}(z, t) = \mathbf{P}^+(z, t) + \mathbf{P}^-(z, t) ,$$

but we include the complex amplitude  $P(z, t) = N |X(z, t)|$ , that is,

$$\begin{aligned} \mathbf{P}^+(z, t) &= \frac{1}{2} \hat{\mathbf{x}}_P(z, t) e^{i[Kz - \nu t - \varphi(z, t)]} \\ &= \frac{1}{2} N(z) |\hat{\mathbf{x}}| X(z, t) e^{i[Kz - \nu t - \varphi(z, t)]} . \end{aligned} \quad (1.26)$$

Here  $N(z)$  is the number of systems per unit volume,  $|\hat{\mathbf{x}}|$  is the dipole moment constant of a single oscillator, and  $X(z, t)$  is a complex dimensionless amplitude that varies little in an optical period or wavelength. In quantum mechanics,  $|\hat{\mathbf{x}}|$  is given by the electric dipole matrix element  $\varphi$ . Since the polarization is real, we have

$$\mathbf{P}^-(z, t) = [\mathbf{P}^+(z, t)]^* . \quad (1.27)$$

It is sometimes convenient to write  $X(z, t)$  in terms of its real and imaginary parts in the form

$$X \equiv U - iV . \quad (1.28)$$

The classical real variables  $U$  and  $V$  have quantum mechanical counterparts that are components of the Bloch vector  $U\hat{\mathbf{e}} + V\hat{\mathbf{e}} + W\hat{\mathbf{e}}$ , as discussed in Sect. 4.3. The slowly-varying amplitude and phase approximation for the polarization is given by

$$\left\{ \frac{\partial U}{\partial t} \right\} = \nu |U| , \quad (1.29)$$

$$\left\{ \frac{\partial V}{\partial t} \right\} = \nu |V| . \quad (1.30)$$

or equivalently by

$$\left\{ \frac{\partial X}{\partial t} \right\} = \nu |X| .$$

We generalize the slowly-varying Maxwell equations (1.17, 1.18) to include the polarization by treating the left-hand side of the wave equation (1.25) as before and substituting (1.26) into the right-hand side of (1.25). Using (1.29, 1.30) to eliminate the time derivatives of  $U$  and  $V$  and equating real imaginary parts separately, we find

$$\frac{\partial E_0}{\partial z} + \frac{1}{c} \frac{\partial E_0}{\partial t} = \frac{K}{2\epsilon} \text{Im}(P) = \frac{K}{2\epsilon} N(z) |V| \quad (1.31)$$

$$E_0 \frac{\partial \varphi}{\partial z} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = -\frac{K}{2\epsilon} \text{Re}P = -\frac{K}{2\epsilon} N(z) |U| \quad (1.32)$$

These two equations play a central role in optical physics and quantum optics. They tell us how light propagates through a medium and specifically how the real and imaginary parts of the polarization act. Equation (1.31) shows that the field amplitude is driven by the *imaginary* part of the polarization. This *in-quadrature* component gives rise to absorption and emission.

Equation (1.32) allows us to compute the phase velocity with which the electromagnetic wave propagates in the medium. It is the real part of the polarization, i.e, the part *in-phase* with the field, that determines the phase velocity. The effects described by this equation are those associated with the *index of refraction* of the medium, such as dispersion and self focusing.

Equations (1.31, 1.32) alone are not sufficient to describe physical problems completely, since they only tell us how a plane electromagnetic wave responds to a given polarization of the medium. That polarization must still be determined. Of course, we know that the polarization of a medium is influenced by the field to which it is subjected. In particular, for atoms or molecules without permanent polarization, it is the electromagnetic field itself that induces their polarization! Thus the polarization of the medium drives the field, while the field drives the polarization of the medium. In general this leads to a description of the interaction between the electromagnetic field and matter expressed in terms of coupled, nonlinear, partial differential equations that have to be solved *self-consistently*. The polarization of a medium consisting of classical simple harmonic oscillators is discussed in Sect. 1.3 and Chap. 2 discusses similar media with anharmonic (nonlinear) oscillators. Two-level atoms are discussed in Chaps. 3–7.

There is no known general solution to the problem, and the art of quantum optics is to make reasonable approximations in the description of the field and/or medium valid for cases of interest. Two general classes of problems reduce the partial differential equations to ordinary differential equations: 1) problems for which the amplitude and phase vary only in time, e.g., in a cavity, and 2) problems for which they vary only in space, i.e., a steady state exists. The second of these leads to Beer's law of absorption,<sup>1</sup> which we consider here briefly. We take the steady-state limit given by

$$\frac{\partial E_0}{\partial t} = 0$$

in (1.31). We further shine a continuous beam of light into a medium that responds linearly to the electric field as described by the slowly-varying complex polarization

$$P = N(z) [ (U - iV) \equiv N(z) ] X = \varepsilon(\chi^r + i\chi^{rr})E_0(z), \quad (1.33)$$

where  $\chi^r$  and  $\chi^{rr}$  are the real and imaginary parts of the linear susceptibility  $\chi$ . This susceptibility is another useful way of expressing the polarization. Substituting the in-quadrature part of  $P$  into (1.31), we obtain

$$\begin{aligned} \frac{dE_0}{dz} &= -\frac{K}{2} \chi^{rr} E_0 \\ &= -\text{Re}\{\alpha\} E_0, \end{aligned} \quad (1.34)$$

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<sup>1</sup> Beer's law is perhaps more accurately called Bouguier-Lambert-Beer's law. We call it Beer's law due to popular usage.

where

$$\sigma = \frac{iKP}{2\varepsilon E_0} = \frac{iKN(z)|X|}{2\varepsilon E_0} = -K \frac{N(z)|}{2\varepsilon E_0} (V + iU) \quad (1.35)$$

is called the complex amplitude absorption coefficient. We use an amplitude absorption coefficient instead of an intensity coefficient to be consistent with coupled-mode equations important for phase conjugation and other nonlinear mode interactions. If  $\chi''$  is independent of  $E_0$ , (1.34) can be readily integrated to give

$$E_0(z) = E_0(0)e^{-\text{Re}\{\sigma\}z}. \quad (1.36)$$

Taking the absolute square of (1.36) gives Beer's law for the intensity

$$I(z) = I(0)e^{-2\text{Re}\{\sigma\}z}. \quad (1.37)$$

We emphasize that this important result can only be obtained if  $\sigma$  is independent of  $I$ , that is, if the polarization (1.33) of the medium responds linearly to the field amplitude  $E_0$ . Chapter 2 shows how to extend (1.33) to treat larger fields, leading to the usual discussion of nonlinear optics. Time dependent fields also lead to results such as (12.27) that differ from Beer's law. For these, (1.33) doesn't hold any more (even in the weak-field limit) if the medium cannot respond fast enough to the field changes. This can lead to effects such as laser lethargy, for which the field is absorbed or amplified according to the law

$$I(z) \propto \exp(-b\sqrt{z}), \quad (1.38)$$

where  $b$  is some constant.

The phase equation (1.32) allows us to relate the in-phase component of the susceptibility to the index of refraction  $n$ . As for the amplitude (1.34), we consider the continuous wave limit, for which  $\partial\varphi/\partial t = 0$ . This gives

$$d\varphi/dz = -K\chi''/2. \quad (1.39)$$

Expanding the slowly varying phase  $\varphi(z) \approx \varphi_0 + z d\varphi/dz$ , we find the total phase factor

$$\begin{aligned} Kz - vt - \varphi &\approx v[(K - d\varphi/dz)z/v - t] - \varphi_0 \\ &= v[(1 + \chi''/2)z/c - t] - \varphi_0 \\ &= v(z/v - t) - \varphi_0. \end{aligned}$$

Noting that the velocity component<sup>2</sup>  $v$  is also given by  $c/n$ , we find the index of refraction (relative to the host medium)

$$n = 1 + \chi''/2. \quad (1.40)$$

<sup>2</sup> Note that the character  $v$ , which represents a speed, is different from the character  $\nu$ , which represents a circular frequency (radians per second).



In coupled-mode problems (see Sects. 2.2, 11.2) and pulse propagation, instead of (1.12) it is more convenient to decompose the electric field in terms of a complex amplitude  $E(z, t) \equiv E_0(z, t) \exp(-i\varphi)$ , that is,

$$E(z, t) = \frac{1}{2} E(z, t) e^{i(Kz - vt)} + \text{c.c.} \quad (1.41)$$

The polarization is then also defined without the explicit  $\exp(i\varphi)$  as

$$P(z, t) = \frac{1}{2} P(z, t) e^{i(Kz - vt)} + \text{c.c.} \quad (1.42)$$

Substituting these forms into the wave equation (1.25) and neglecting small terms like  $\partial^2 E / \partial t^2$ ,  $\partial^2 P / \partial t^2$ , and  $\partial P / \partial t$ , and equating the coefficients of  $e^{i(Kz - vt)}$  on both sides of the equation, we find the slowly-varying Maxwell's equation

$$\frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} = i \frac{K}{2\varepsilon} P \quad (1.43)$$

Note that in equating the coefficients of  $e^{i(Kz - vt)}$ , we make use of our assumption that  $P(z, t)$  varies little in a wavelength. Should it vary appreciably in a wavelength due, for example, to a grating induced by an interference fringe, we would have to evaluate a projection integral as discussed for standing wave interactions in Sect. 5.3.

In a significant number of laser phenomena, the plane-wave approximation used in this chapter is inadequate. For these problems, Gaussian beams may provide a reasonable description. A simple derivation of the Gaussian beam as a limiting case of a spherical wave  $\exp(iKr)/r$  is given in Sect. 7.7.

## Group velocity

The preceding discussion introduced the velocity  $v = c/n$ , which is the velocity at which the phase of a monochromatic wave of frequency  $\nu$  propagates in a medium with index of refraction  $n(\nu)$ , or phase velocity. Consider now the situation of two plane monochromatic waves of same amplitude  $E$  that differ slightly in frequency and wave number,

$$\begin{aligned} E(z, t) &= E e^{i[(k_0 + \Delta k)z - (\nu_0 + \Delta \nu)t]} + E e^{i[(k_0 - \Delta k)z - (\nu_0 - \Delta \nu)t]} \\ &= 2E e^{i(k_0 z - \nu_0 t)} \cos \left[ \Delta k z - \Delta \nu t \right] \end{aligned}$$

When adding a group of waves with a small spread of wave numbers and frequencies about  $k_0$  and  $\nu_0$ , we find similarly that the total field consists of a carrier wave with phase velocity  $v = c/n$  and group velocity

$$v_g = \frac{dv}{dk} \quad (1.44)$$

In case the absorption of light at the frequency  $\nu_0$  is sufficiently weak to be negligible,  $\nu_g$  can be taken to be real and with  $k = \nu n(\nu)/c$  we find readily

$$\nu_g = \frac{d\nu}{dk} = \frac{c}{(n + \nu dn/d\nu)_{\nu_0}}. \quad (1.45)$$

We observe that in regions of “normal dispersion”,  $dn/d\nu > 0$ , the group velocity is less than the phase velocity. However, the situation is reversed in regions of “anomalous dispersion”,  $dn/d\nu < 0$ . Indeed  $\nu_g$  can even exceed  $c$  in this region. This has been the origin of much confusion in the past, in particular it has been mentioned that this could be in conflict with special relativity. This, however, is not the case. This is incorrect, because the group velocity is not in general a signal velocity. This, as many other aspects of “fast light” and “slow light,” is discussed very clearly in Milonni (2005).

Chapter 12 discusses how quantum interference effects such as electromagnetically induced transparency can be exploited to dramatically manipulate the group velocity of light, resulting in particular in the generation of “slow light.”

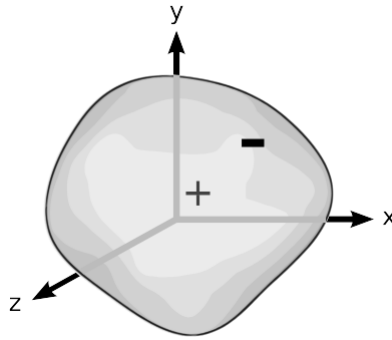
## Linear Dipole Oscillator

As a simple and important example of the interaction between electromagnetic waves and matter, let us consider the case of a medium consisting of classical damped linear dipole oscillators. As discussed in Chap. 3, this model describes the absorption by quantum mechanical atoms remarkably well. Specifically we consider a charge (electron) cloud bound to a heavy positive nucleus and allowed to oscillate about its equilibrium position as shown in Fig. 1.1. We use the coordinate  $x$  to label the deviation from the equilibrium position with the center of charge at the nucleus. For small  $x$  it is a good approximation to describe the motion of the charged cloud as that of a damped simple harmonic oscillator subject to a sinusoidal electric field. Such a system obeys the Abraham-Lorentz equation of motion

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega^2 x(t) = \frac{e}{m} E(t), \quad (1.46)$$

where  $\omega$  is the natural oscillation frequency of the oscillator, and the dots stand for derivatives with respect to time. Note that since oscillating charges radiate, they lose energy. The end of this section shows how this process leads naturally to a damping constant  $\gamma$ . Quantum mechanically this decay is determined by spontaneous emission and collisions.

The solution of (1.44) is probably known to the reader. We give a derivation below that ties in carefully with the corresponding quantum mechanical treatments given in Chaps. 4, 5. Chapter 2 generalizes (1.44) by adding nonlinear forces proportional to  $x^2$  and  $x^3$  [see (2.1)]. These forces lead to



**Fig. 1.1.** Negative charge cloud bound to a heavy positive nucleus by Coulomb attraction. We suppose that some mysterious forces prevents the charge cloud from collapsing into the nucleus

coupling between field modes producing important effects such as sum and difference frequency generation and phase conjugation. As such (1.44) and its nonlinear extensions allow us to see many “atom”-field interactions in a simple classical context before we consider them in their more realistic, but complex, quantum form.

We suppose the electric field has the form

$$E(t) = \frac{1}{2} E_0 e^{-i\nu t + \text{c.c.}}, \quad (1.47)$$

where  $E_0$  is a constant real amplitude. In general the phase of  $x(t)$  differs from that of  $E(t)$ . This can be described by a complex amplitude for  $x$ , that is,

$$x(t) = \frac{1}{2} x_0 X(t) e^{-i\nu t + \text{c.c.}}, \quad (1.48)$$

where  $X(t)$  is the dimensionless complex amplitude of (1.26). In the following we suppose that it varies little in the damping time  $1/\gamma$ , which is a much more severe approximation than the SVAP. Our problem is to find the steady-state solution for  $X(t)$ .

As in the discussion of (1.33, 1.34), we substitute (1.45, 1.46) into (1.44), neglect the small quantities  $\dot{X}$  and  $\gamma X$ , and equate positive frequency components. This gives

$$\dot{X} = -[\gamma + i(\omega^2 - \nu^2)/2\nu] X + \frac{ieE_0}{2\nu m x_0}. \quad (1.49)$$

In steady state ( $\dot{X} = 0$ ), this gives the amplitude

$$X = \frac{ieE_0/2\nu m x_0}{\gamma + i(\omega^2 - \nu^2)/2\nu}, \quad (1.50)$$

and hence the displacement

$$x(t) = \frac{i e E_0}{2 \gamma + i(\omega^2 - \nu^2)/2\nu} e^{-i\nu t} + \text{c.c.} \quad (1.51)$$

We often deal with the *near resonance*, that is, the situation where  $|\nu - \omega| \ll \omega, \nu$ . For this case we can make the classical analog of the *rotating-wave approximation* defined in Sect. 3.2. Specifically we approximate  $\omega^2 - \nu^2$  by

$$\omega^2 - \nu^2 \approx 2\nu(\omega - \nu). \quad (1.52)$$

This reduces (1.48, 1.49) to

$$X = \frac{i e E_0 / 2 \nu m x_0}{\gamma + i(\omega - \nu)}, \quad (1.53)$$

$$x(t) = \frac{i e E_0}{2 \gamma + i(\omega - \nu)} e^{i\nu t} + \text{c.c.} \quad (1.54)$$

Equation (1.52) shows that in steady state the dipole oscillates with the same frequency as the driving field, but with a different phase. At resonance ( $\nu = \omega$ ), (1.52) reduces to

$$x(t, \nu = \omega) = \frac{e E_0}{2 m \nu \gamma} \sin \nu t, \quad (1.55)$$

that is, the dipole lags by  $\pi/2$  behind the electric field (1.45), which oscillates as  $\cos \nu t$ . The corresponding polarization of the medium is  $P = N e x(t)$ , where  $N$  is the number of oscillators per unit volume. Substituting this along with (1.52) into (1.35), we find the complex amplitude Beer's law absorption coefficient

$$\alpha = K \frac{N e^2 \gamma}{2 \epsilon \gamma 2 m \nu \gamma + i(\omega - \nu)}$$

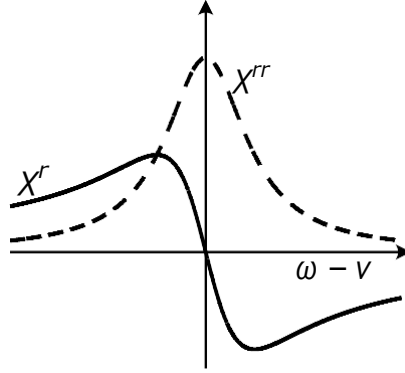
or

$$\alpha = \frac{a_0 \gamma [\gamma - i(\omega - \nu)]}{\gamma^2 + (\omega - \nu)^2}, \quad (1.56)$$

where the resonant absorption coefficient  $a_0 = K N e^2 / 4 \epsilon \gamma m \nu$ . The real part of this expression shows the Lorentzian dependence observed in actual absorption spectra (see Fig. 1.2). The corresponding quantum mechanical absorption coefficient of (5.29) differs from (1.54) in three ways:

1.  $\gamma^2 + (\omega - \nu)^2$  is replaced by  $\gamma^2(1 + I) + (\omega - \nu)^2$  }
  2.  $N$  becomes negative for gain media
  3.  $e^2/2m\nu$  is replaced by  $\mu/k$  }
- (1.57)

For weak fields interacting with absorbing media, only the third of these differences needs to be considered and it just defines the strength of the dipole moment being used. Hence the classical model mirrors the quantum mechanical one well for linear absorption (for a physical interpretation of this result, see Sect. 3.2).



**Fig. 1.2.** Absorption (Lorentzian bell shape) and index parts of the complex absorption coefficient of (1.54)

Identifying the real and imaginary parts of (1-47) and using (1.33), we obtain the equations of motion for the classical Bloch-vector components  $U$  and  $V$

$$\dot{U} = -(\omega - \nu)V - \gamma U, \quad (1.58)$$

$$\dot{V} = (\omega - \nu)U - \gamma V - eE_0/2m\nu x_0. \quad (1.59)$$

Comparing (1.57) with (4.49) (in which  $\gamma = 1/T_2$ ), we see that the  $E_0$  term is multiplied by  $-W$ , which is the third component of the Bloch vector. This component equals the probability that a two-level atom is in the upper level minus the probability that it is in the lower level. Hence we see that the classical (1.57) is reasonable as long as  $W \ll 1$ , i.e., so long as the atom is in the lower level.

From the steady-state value of  $X$  given by (1.51), we have the steady-state  $U$  and  $V$  values

$$U = \frac{eE_0}{2m\nu x_0} \frac{\omega - \nu}{\gamma^2 + (\omega - \nu)^2} \quad (1.60)$$

and

$$V = -\frac{eE_0}{2m\nu x_0} \frac{\gamma}{\gamma^2 + (\omega - \nu)^2}. \quad (1.61)$$

Since (1.44) is linear, once we know the solution for the single frequency field (1.45), we can immediately generalize to a multifrequency field simply by taking a corresponding superposition of single frequency solutions. The various frequency components in  $x(t)$  oscillate independently of one another. In contrast the nonlinear media in Chap. 2 and later chapters couple the modes. Specifically, consider the multimode field

$$E(z, t) = \frac{1}{2} \sum_n E_n(z) e^{i(K_n z - \nu_n t)} + \text{c. c.}, \quad (1.62)$$

where we allow the field amplitudes to be slowly varying functions of  $z$  and to be complex since they do not in general have the same phases. The solution for the oscillator displacement  $x(t)$  at the position  $z$  is a superposition of solutions like (1.46), namely,

$$x(t) = \frac{1}{2} \sum_n x_{0n} X_n e^{i(K_n z - \nu_n t)} + \text{c. c.} , \quad (1.63)$$

where mode  $n$ 's oscillator strength is proportional to  $x_{0n}$  and the coefficients

$$X_n = \frac{eE_n/mx_{0n}}{\omega^2 - \nu_n^2 - 2i\nu_n\gamma} . \quad (1.64)$$

Here we don't make the resonance approximation of (1.50), since some of the modes may be off resonance. The steady-state polarization  $P(z, t)$  of a medium consisting of such oscillators is then given by

$$P(z, t) = \frac{1}{2} \sum_n P_n(z) e^{i(K_n z - \nu_n t)} + \text{c. c.} , \quad (1.65)$$

where  $P_n(z)$  is given by  $N(z)ex_{0n}X_n$ . In Sect. 2.1, we find that higher-order terms occur when nonlinearities are included in the equation of motion (1.44). These terms couple the modes and lead to anharmonic response. Finally, we note that the multimode field (1.60) and the polarization (1.63) have the same form in the unidirectional ring laser of Chap. 7, except that in a high- $Q$  cavity the mode amplitudes  $E_n$  and polarization components  $P_n$  are functions of  $t$ , rather than  $z$ .

## Radiative Damping

We now give a simple approximate justification for the inclusion of a damping coefficient  $\gamma$  in (1.44). As a charge oscillates it radiates electromagnetic energy and consequently emits a "self-field"  $E_s$ . We need to find the influence of this self-field back on the charge's motion in a self-consistent fashion. We find that the main effect is the exponential damping of this motion as given by (1.44). Specifically, we consider the equation governing the charge's motion under the influence of the self-field  $E_s$ :

$$m \ddot{x} + \omega^2 x = \frac{e}{m} E_s , \quad (1.66)$$

which is just Newton's law with the Lorentz force

$$\mathbf{F}_s = e(\mathbf{E}_s + \mathbf{v} \times \mathbf{B}_s) \quad (1.67)$$

in the limit of small charge velocities ( $v \ll c$ ), where the magnetic part of the Lorentz force may be neglected.

While we don't know the explicit form of  $E_s$ , we can calculate its effects using the conservation of energy. We evaluate the force  $\mathbf{F}_{\text{rad}}$  of the radiating charge by equating the work it depends on the charge (during a time interval long compared to the optical period  $1/\omega$ ) to minus the energy radiated by the charge during that time

$$\int_t^{t+\Delta t} \mathbf{F}_{\text{rad}} \cdot \mathbf{v} dt' = - \int_t^{t+\Delta t} (\text{radiated power}) dt'. \quad (1.68)$$

To calculate the radiated power, we note that the instantaneous electromagnetic energy flow is given by the Poynting vector

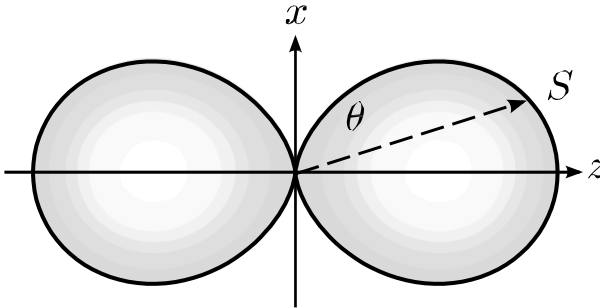
$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E}_s \times \mathbf{B}_s, \quad (1.69)$$

where for simplicity we suppose that the “host medium” is the vacuum. We note that the electric field radiated in the far field of the dipole is

$$\mathbf{E}_s(\mathbf{R}, t) = \frac{e}{4\pi\epsilon^0 c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}})}{R} \quad t=R/c \quad (1.70)$$

as shown in Fig. 1.3. The corresponding magnetic field is  $\mathbf{B}_s(\mathbf{R}, t) = c^{-1} \mathbf{n} \times \mathbf{E}_s(\mathbf{R}, t)$ . In both expressions the dipole acceleration  $\dot{\mathbf{v}}$  is evaluated at the retarded time  $t - R/c$  and  $\mathbf{n}$  is the unit vector  $\mathbf{R}/R$ . Inserting these expressions into (1.66), we find the Poynting vector [ Jackson (1999)]

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0 c} (\mathbf{E}_s \cdot \mathbf{E}_s) \mathbf{n} \\ &= \frac{e^2}{16\pi^2 \epsilon_0^2 c^4} \frac{1}{\mu_0 c R^2} (\mathbf{n} \times \dot{\mathbf{v}})^2 \mathbf{n} \\ &= \frac{e^2 \dot{v}^2 \sin^2 \theta}{16\pi^2 \epsilon^0 c^3 R^2} \mathbf{n}. \end{aligned} \quad (1.71)$$



**Fig. 1.3.** Butterfly pattern given by (1.69) and emitted by an oscillating dipole. The vector gives the direction and relative magnitude of the Poynting vector  $\mathbf{S}$  as a function of  $\theta$ .

The total power radiated is given by integration of  $\mathbf{S}$  over a sphere surrounding the charge. Noting that

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^3 \theta = -2\pi \int_1^{-1} d(\cos \theta)(1 - \cos^2 \theta) = 8\pi/3, \quad (1.72)$$

we find

$$\mathbf{S} \cdot d\mathbf{a} = \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 c^3} \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}, \quad (1.73)$$

which is the Larmor power formula for an accelerated charge. We now substitute (1.71) into (1.66) and integrate by parts. We encounter the integral

$$\int_t^{t+\Delta t} dt \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} = \dot{\mathbf{v}} \cdot \mathbf{v} \Big|_t^{t+\Delta t} - \int_t^{t+\Delta t} dt \mathbf{v} \cdot \ddot{\mathbf{v}}.$$

Since  $\mathbf{v}$  and its derivatives are periodic, the constant of integration on the right hand side has a maximum magnitude, while the integrals continue to increase as  $\Delta t$  increases. Hence the constant can be dropped. Equating the integrands, we find the radiation force

$$\mathbf{F}_{\text{rad}} = \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 c^3} \ddot{\mathbf{v}}. \quad (1.74)$$

A more detailed analysis of this problem is given in Sect. 19.3 of Jackson (1999), where the infinities associated with point-like charges are also discussed.

Assuming that the radiative damping is sufficiently small that the motion of the dipole remains essentially harmonic, (1.72) yields

$$\mathbf{F}_{\text{rad}} = m\ddot{\mathbf{x}} = -\frac{2}{3} \frac{e^2 \omega^2}{4\pi\epsilon_0 c^3} \mathbf{v}, \quad (1.75)$$

which indicates that radiation reaction acts as a friction on the motion of the charge. This implies a damping rate constant

$$\gamma = \frac{1}{4\pi\epsilon_0} \frac{1}{3} \frac{e^2 \omega^2}{c^3 m} = \frac{1}{3} \frac{\omega^2 r_0}{c}, \quad (1.76)$$

where the classical radius of the electron is

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} \approx 2.8 \times 10^{-15} \text{ meters}. \quad (1.77)$$

For 1  $\mu\text{m}$  radiation,  $\gamma = 2\pi \times 1.8 \text{ MHz}$ , which is in the range of decay values found in atoms. In cgs units the  $4\pi\epsilon_0$  in (1.74, 1.75).

With the replacement of  $e^2/2m\mathbf{v}$  by  $\wp^2/\hbar$ , see (1.55), the classical decay rate (1.74) gives half the quantum mechanical decay rate (14.60). Here  $\wp$  is the reduced dipole matrix element between the upper and lower level transition.



The other half of the decay rate is contributed by the effects of vacuum fluctuations missing in a classical description. Note that in both the classical and quantum mechanical cases, an  $\omega^2$  term appears. In the quantum case, this term results from the density of states of free space (14.46), while for the classical case it comes from the acceleration of the electron. In some sense the density of states for the field reflects the fact that the field itself is radiated by accelerating, oscillating charges. In free space the charge responsible for this field is the bound electron itself, radiating a field that acts back on the charge and causes it to emit radiation until no more downward transitions are possible. For further discussion, see Milonni (1986, 1984, 1994).

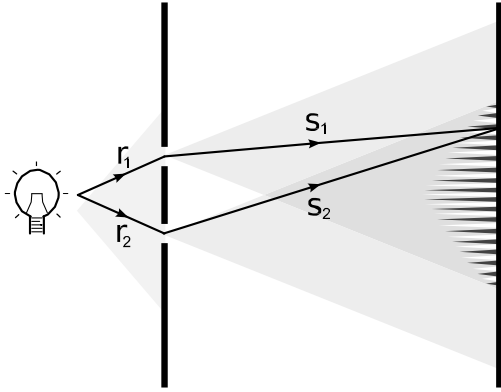
## Coherence

Coherence plays a central role in modern physics. It is very hard to find a single domain of physics where this concept is not applied. In this book we use it a great deal, speaking of coherent light, coherent transients, coherent propagation, coherent states, coherent excitation, etc. Just what is coherent? The answer typically depends on whom you ask! In a very general sense, a process is coherent if it is characterized by the existence of some well-defined deterministic phase relationship, or in other words, if some phase is not subject to random noise. This is a very vague definition, but general enough to encompass all processes usually called “coherent”. In this section and Sect. 13.5 we consider the coherence of classical light. Chapters 4, 12 discuss coherence in atomic systems.

The classic book by Born and Wolf (1970) gives a discussion of coherent light in pre-laser terms. With the advent of the laser, a number of new effects have been discovered that have caused us to rethink our ideas about coherent light. In addition, the Hanbury Brown-Twiss experiment, which had nothing to do with lasers, plays an important role in this rethinking. Our discussion is based on the theory of optical coherence as developed by R. Glauber and summarized in his Les Houches lectures (1965).

We start with the famous Young double-slit experiment which shows how coherent light passing through two slits interferes giving a characteristic intensity pattern on a screen (see Fig. 1.4). Before going into the details of this experiment, we need to know how the light intensity is measured, either on a screen or with a photodetector. Both devices work by absorbing light. The absorption sets up a chemical reaction in the case of film, and ionizes atoms or lifts electrons into a conduction band in the cases of two kinds of photodetectors. Section 13.5 shows by a quantum-mechanical analysis of the detection process that these methods measure  $|E^+(\mathbf{r}, t)|^2$ , rather than  $|E(\mathbf{r}, t)|^2$ . This is why we performed the decomposition in (1.9).

Returning to Young’s double-slit experiment, we wish to determine  $E^+(\mathbf{r}, t)$ , where  $\mathbf{r}$  is the location of the detector.  $E^+(\mathbf{r}, t)$  is made up of two components, each coming from its respective slit



**Fig. 1.4.** Young double-slit experiment illustrating how coherent light can interfere with itself

$$E^+(\mathbf{r}, t) = E^+(\mathbf{r}_1, t_1) + E^+(\mathbf{r}_2, t_2) , \quad (1.78)$$

where  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are locations of the slits and  $t_1$  and  $t_2$  are the retarded times

$$t_{1,2} = t - s_{1,2}/c \quad (1.79)$$

$s_1$  and  $s_2$  being the distances between the slits and the detector. From (1.76), the intensity at the detector is given by

$$\begin{aligned} |E^+(\mathbf{r}, t)|^2 &= |E^+(\mathbf{r}_1, t_1)|^2 + |E^+(\mathbf{r}_2, t_2)|^2 \\ &\quad + 2 \operatorname{Re}[E^-(\mathbf{r}_1, t_1)E^+(\mathbf{r}_2, t_2)] , \end{aligned} \quad (1.80)$$

where we have made use of (1.9).

In general the light source contains noise. To describe light with noise we use a statistical approach, repeating the measurement many times and averaging the results. Mathematically this looks like

$$\begin{aligned} \langle |E^+(\mathbf{r}, t)|^2 \rangle &= \langle |E^+(\mathbf{r}_1, t_1)|^2 \rangle + \langle |E^+(\mathbf{r}_2, t_2)|^2 \rangle \\ &\quad + 2 \operatorname{Re} \langle E^-(\mathbf{r}_1, t_1)E^+(\mathbf{r}_2, t_2) \rangle , \end{aligned} \quad (1.81)$$

where the brackets  $\langle \dots \rangle$  stands for the ensemble average. Introducing the first-order correlation function

$$G^{(1)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) \equiv \langle E^-(\mathbf{r}_1, t_1)E^+(\mathbf{r}_2, t_2) \rangle , \quad (1.82)$$

we rewrite (1.79) as

$$\begin{aligned} \langle |E^+(\mathbf{r}, t)|^2 \rangle &= G^{(1)}(\mathbf{r}_1 t_1, \mathbf{r}_1 t_1) + G^{(1)}(\mathbf{r}_2 t_2, \mathbf{r}_2 t_2) \\ &\quad + 2 \operatorname{Re} G^{(1)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) . \end{aligned} \quad (1.83)$$

$G^{(1)}(\mathbf{r}_i t_i, \mathbf{r}_i t_i)$  is clearly a real, positive quantity, while  $G^{(1)}(\mathbf{r}_i t_i, \mathbf{r}_j t_j)$  is in general complex.