$\boxed{\text{Product Rule:}}$

$$p(A, B) = \text{“joint probability of } both\ A\ and\ B\text{”}$$
$$= p(A|B)p(B)$$

or equivalently,

$$= p(B|A)p(A)$$

Clearly, in case A and B are *independent* events, they are not conditionalized on each other and so

$$p(A|B) = p(A)$$
$$\text{and } p(B|A) = p(B),$$

in which case their joint probability is simply $p(A, B) = p(A)p(B)$.

$\boxed{\text{Sum Rule:}}$

If event A is conditionalized on a number of other events B, then the total probability of A is the sum of its joint probabilities with all B:

$$p(A) = \sum_B p(A, B) = \sum_B p(A|B)p(B)$$

From the Product Rule and the symmetry that $p(A, B) = p(B, A)$, it is clear that $p(A|B)p(B) = p(B|A)p(A)$. Bayes' Theorem then follows:

<u>Bayes' Theorem:</u>

$$p(B|A) = \frac{p(A|B)p(B)}{p(A)}$$

The importance of Bayes' Rule is that it allows us to reverse the conditionalizing of events, and to compute $p(B|A)$ from knowledge of $p(A|B)$, $p(A)$, and $p(B)$. Often these are expressed as *prior* and *posterior* probabilities, or as the conditionalizing of hypotheses upon data.

Question 1:

An urn B_1 contains 2 white and 3 black balls and another urn B_2 contains 3 white and 4 black balls. One urn is selected at random and a ball is drawn from it. If the ball drawn is found black, find the probability that the urn chosen was B_1.

Solution:

Step1:

Let E_1, E_2 denote the events of selecting urns B_1 and B_2 respectively.

Then $P(E_1) = P(E_2) = 1/2$.

Then we have to find P(E1/B).

Step 2:

By hypothesis $P(B/E_1) = 3/5$ and $P(B/E_2) = 4/7$

By Bayes theorem

$P(E_1/B) = P(E1)P(B/E1) \,/\, P(E1)P(B/E1)+P(E2)P(B/E2)$

$= (1/2*3/5)/\ (1/2*3/5+1/2*4/7)$

$= 21/41$

Question 2:

The urns contain 6 green, 4 black; 4 green, 6 black and 5 green, 5 black balls respectively. Randomly selected an urn and a ball is drawn from it. If the ball drawn is Green, find the probability that it is drawn from the first urn.

Solution:

Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = urn first is chosen, E_2 = urn second is chosen,

E_3 = urn third is chosen, and A = ball drawn is Green.

Step 1:

Since there are three urns and one of the three urns is chosen at random, therefore

$P(E_1) = P(E_2) = P(E_3) = 1/3$

If E_1 is already occurred, then urn first has been chosen which contains 6 Green and 4 Black balls. The probability of drawing a green ball from it is 6/10.

So, $P(A/E_1) = 6/10$

Similarly, we have $P(A/E_2) = 4/10$ and $P(A/E3) = 5/10$

Step 2:

We are required to find P(E1/A) , i.e. given that the ball drawn is Green , what is the probability that it is drawn from the first urn. By Bayes' theorem, we have

$P(E_1/A) = P(E1)P(A/E1)/$
$P(E1)P(A/E1)+P(E2)P(A/E2)+P(E3)P(A/E3)$

$= (1/3*6/10)/\ \ (1/3*6/10)+(1/3*4/10)\ +(1/3*5/10)$

$= 6/15$

$= 2/5$

=> $P(E_1/A) = 2/5$.

Question 3: A is known to tell the truth in 5 cases out of 6 and he states that a white ball was drawn from a bag containing 8 blacks and 1 white ball. Find the probability that the white ball was drawn.

Solution:

Step:1

Let W denote the event that A draws a white ball and T the event that A speaks truth. In the usual notations, we are given that

$P(W) = 1/9$,

$P(T/W) = 5/6$

=> $P(\bar{W}) = 1 – 1/9 = 8/9$

and $P(T/\bar{W}) = 1 – 5/6 = 1/6$

Step 2:

Using Baye's theorem , the required probability is given by

$P(W/T)$

$= P(T/W)P(W) \; / \; P(T/W)P(W) + P(T/\bar{W})P(\bar{W})$

$= 5/6*1/9 \; / \; 5/6*1/9 + 1/6*8/9$

$= 5/13$

=> $P(W/T) = 5/13$.

Example, flipping a fair coin once will give us events h and t each with probability 1/2, and thus a single flip of a coin gives us

$-\log_2 (1/2) = 1$ bit of information (whether it comes up h or t).

Flipping a fair coin n times (or, equivalently, flipping n fair coins) gives us

$-\log_2((1/2)^n) = \log_2(2^n) = n * \log_2(2) = n$ bits of information. We could enumerate a sequence of 25 flips as, for example:

hthhtththhhthttththhhthtt or, using 1 for h and 0 for t, the 25 bits

1011001011101000101110100:

We thus get the nice fact that n flips of a fair coin gives us n bits of information, and takes n binary digits to specify. That these two are the same reassures us that we have done a good job in our definition of our information measure.