System of Linear Equation

Definition 1

Let the system of linear equation as

$$\left.\begin{matrix} a_{11}x_{1,}\ a_{12}x_{2\ldots}\ ,a_{1n}x_n = b_1 \\ a_{21}x_{1,}\ a_{22}x_{2\ldots}\ ,a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{m1}x_{1,}\ a_{m2}x_{2\ldots}\ ,a_{mn}x_n = b_m \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

Can put the apove system in matrix form as:-

$$\begin{pmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

Or

$AX=B$, ……………………………………………………………….. (8)

$$A=\begin{pmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{pmatrix},\ B=\begin{pmatrix} b_1 \\ b_2 \\ - \\ - \\ b_m \end{pmatrix},\ \text{and } X=\begin{pmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{pmatrix}$$

Where $A=mxn$, matrix, $a_{11,}\ a_{12\ldots}$, a_{mn} are constant, $X= nx1$, $B = mx1$and, $b_{1,}\ b_{2\ldots}$, b_m, are constant $x_{1,}\ x_{2\ldots}\ x_n$, variable.

Now we study the following methods {Cramer's Rule, Inverse Matrices, and Elimination Method}

Cramer's Rule

To solve the system (8) by Cramer's Rule. Find determinate of A (| A|) such that $|A| \neq 0$.
Let

$$|A| = D = \begin{vmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & a_{mn} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & - & - & - & a_{1n} \\ b_2 & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ b_m & a_{m1} & - & - & - & a_{mn} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & - & - & - & a_{1n} \\ a_{21} & b_2 & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & b_m & - & - & - & a_{mn} \end{vmatrix}, \ldots, D_n = \begin{vmatrix} a_{11} & a_{12} & - & - & - & b_1 \\ a_{21} & a_{22} & - & - & - & b_2 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m1} & - & - & - & b_m \end{vmatrix},$$

To solve system (8), we must find unknown $x_1, x_2 \ldots x_n$ as

$$x_1 = \frac{D_1}{D}, \; x_2 = \frac{D_2}{D}, \ldots x_n = \frac{D_n}{D}.$$

د-Solution of Linear Equations by using Inverse Matrices

To solve the system (8) by **using Inverse Matrices** Find determinate of A (| A|) such that $|A| \neq 0$.
Or
AX=B,
Turing to the relation between the solution of linear equation and matrix inversion multiplying both sides by A^{-1} thus
A^{-1} [AX=B]
$A^{-1} AX = A^{-1} B$.
$X = A^{-1} B$.
This equation gives the values of the entire unknown X by a simple multiplication of matrix A by inverse of it matrix. As see in the following example

Example12

Use the matrix inversion method; find the values of (x_1, x_2, x_3) for the following set of linear algebraic equations:-

$$\left.\begin{aligned} 3x_1 - 6x_2 + 7x_3 &= 3 \\ 4x_1 \qquad - 5x_3 &= 3 \\ 5x_1 - 8x_2 + 6x_3 &= -4 \end{aligned}\right\} \ldots\ldots\ldots\ldots (9)$$

Solution

Put the system (9) in the following matrix form as
AX=B,

$$\begin{pmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

Where| A|

$$|A| = \begin{vmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{vmatrix} = 462 \neq 0.$$

We can find the inverse matrix of A (A^{-1}), by any method.

$$\therefore A^{-1} = \begin{pmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.04 & -0.36 \end{pmatrix}$$, now we can see the following

A^{-1} [AX=B]

$A^{-1}AX = A^{-1}B$.

$X = A^{-1}B$.

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.04 & -0.36 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$, which gives the solution of system as $x_1 = 2$, $x_2 = 4$, $x_3 = -4$.

3--Gauss Elimination Method-

We can use Gauss Elimination Method to solve the system of linear equation in (8), as see in the following example

Example 1: Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5. \end{aligned}$$

Solution: We carry out the elimination procedure on both the system of equations and the correspondi augmented matrix, simultaneously. In general only one set of reductions is necessary, and the latter (dealing with matrices only) is preferable because of the simplified notation.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned} \qquad \begin{pmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{pmatrix}$$

$R_2 + 2R_1 \rightarrow R_2$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ 3x_2 &= 9 \end{aligned} \qquad \begin{pmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{pmatrix}$$

Multiply Row 2 by 1/3. $(\frac{1}{3}. R_2)$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned} \qquad \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

so

$X_2=3$ ……….(1)

$x_1+5x_2=7$…………(2)

from (2) $x_1+5(3)=7$ then $x_1=-8$

Example 2: Use Gaussian elimination to solve the system of linear

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

$$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & -8 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Swap Row 1 and Row 2. $R_1 \longrightarrow R_2$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ -x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Add Row 1 to Row 3. $R_3 + R_1 \longrightarrow R_3$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & & 2x_2 & + & x_3 & = & -8 \\ & - & x_2 & - & x_3 & = & 3 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

Swap Row 2 and Row 3. $R_2 \longrightarrow R_3$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & 2x_2 & + & x_3 & = & -8 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{pmatrix}$$

Add twice Row 2 to Row 3. $R_3 + 2R_2$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ & - & x_2 & - & x_3 & = & 3 \\ & & & - & x_3 & = & -2 \end{array} \qquad \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

So

$X_3 = 2$……………..(1)

$-X_2 - x_3 = 3$ …………..(2)

$X_1 - 2x_2 - 3x_3 = 0$………(3)

From (2) we have $x_2 = -5$

From (3) we have $x_1=-4$

Example 3: Use Gaussian elimination to solve the system of linear equations

$$
\begin{aligned}
x_1 - 2x_2 - 6x_3 &= 12 \\
2x_1 + 4x_2 + 12x_3 &= -17 \\
x_1 - 4x_2 - 12x_3 &= 22.
\end{aligned}
$$

Solution: In this case, we convert the system to its corresponding augmented matrix, perform the necessary row operations on the matrix alone, and then convert back to equations at the end to identif the solution.

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & - & 6x_3 & = & 12 \\
2x_1 & + & 4x_2 & + & 12x_3 & = & -17 \\
x_1 & - & 4x_2 & - & 12x_3 & = & 22
\end{array}
\qquad
\begin{pmatrix}
1 & -2 & -6 & 12 \\
2 & 4 & 12 & -17 \\
1 & -4 & -12 & 22
\end{pmatrix}
$$

$R_2 - 2R_1 \rightarrow R_2$

$R_3 - R_1 \rightarrow R_3$

$$
\begin{pmatrix}
1 & -2 & -6 & 12 \\
0 & 8 & 24 & -41 \\
0 & -2 & -6 & 10
\end{pmatrix}
$$

Swap Row 2 and Row 3. $R_2 \longleftrightarrow R_3$

$$
\begin{pmatrix}
1 & -2 & -6 & 12 \\
0 & -2 & -6 & 10 \\
0 & 8 & 24 & -41
\end{pmatrix}
$$

Add 4 times Row 2 to Row 3. $R_3 + 4R_2$

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 6x_3 & = & 12 \\ & - & 2x_2 & - & 6x_3 & = & 10 \\ & & & & 0 & = & -1 \end{array}$$

Since the final equation

$$0 = -1$$

cannot be satisfied, this system has no solutions.