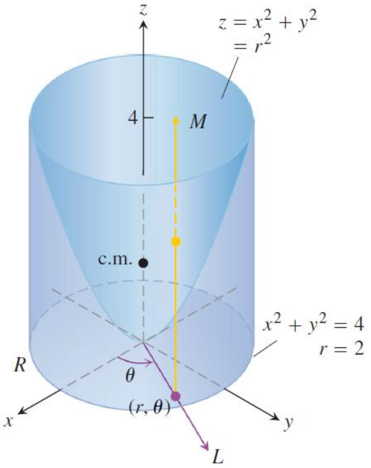
Triple Integration in Cylindrical Coordinates

To integrate continuous function $f(r, \theta, z)$ over a region given by :

$z_1(r,\theta) \leq z \leq z_2(r,\theta)$,
$r_1(\theta) \leq r \leq r_2(\theta)$,
$\theta_1 \leq \theta \leq \theta_2$

$$\boxed{\iiint_D f(r,\theta,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{r=r_1(\theta)}^{r=r_2(\theta)} \int_{z=z_2(r,\theta)}^{z=z_1(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.}$$

EXAMPLE 15: Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$ bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.



Solution:

$x^2 + y^2 = 4$, $(x^2 + y^2 = r^2)$ $\Rightarrow$ $r^2 = 4$ $\Rightarrow$ $r = 2$
$z = x^2 + y^2$ $\Rightarrow$ $z = r^2$

$z_1(r,\theta) \leq z \leq z_2(r,\theta)$: $0 \leq z \leq r^2$

$r_1(\theta) \leq r \leq r_2(\theta)$: $0 \leq r \leq 2$

$\theta_1 \leq z \leq \theta_2$: $0 \leq \theta \leq 2\pi$

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^5}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}\,d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[z\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

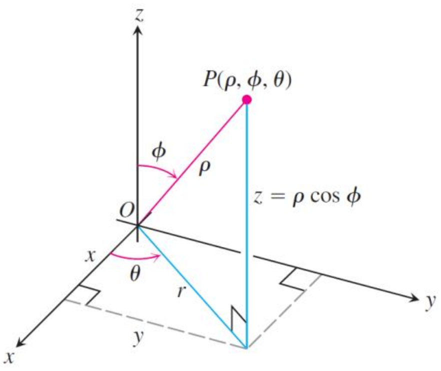
and the centroid is (0, 0, 4/3). Notice that the centroid lies outside the solid.

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance

Spherical coordinates represent a point P in space by ordered triples in which

1. ρ : Sphere radius
2. ϕ : angle between ρ and positive z-axis ($0 \le \phi \le \pi$).
3. θ : angle between ρ and *x*-axis ($0 \le \theta \le 2\pi$).



To integrate continuous function *f(r, θ, z)* over a region given by :

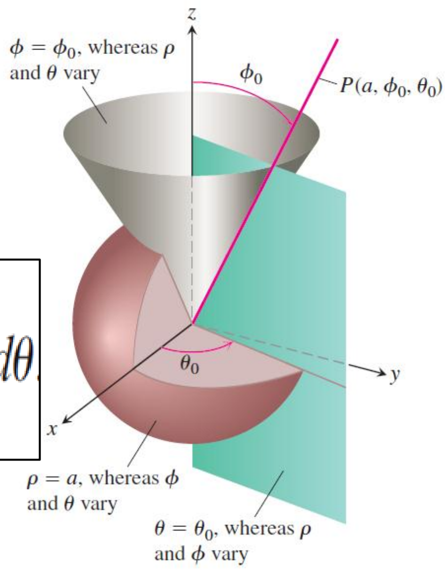
$$\rho_1(\phi,\theta) \le \rho \le \rho_2(\phi,\theta)\ ,$$

$$\phi_1(\theta) \le \phi \le \phi_2(\theta),$$

$$\theta_1 \le \theta \le \theta_2$$

$$\iiint_D f(\rho, \phi, \theta)\, dV =$$

$$\boxed{\int_{\theta=\theta_1}^{\theta=\theta_2} \int_{\phi=\phi_1(\theta)}^{\phi=\phi_2(\theta)} \int_{\rho=\rho_1(\phi,\theta)}^{\rho=\rho_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$



Coordinate Conversion Formulas

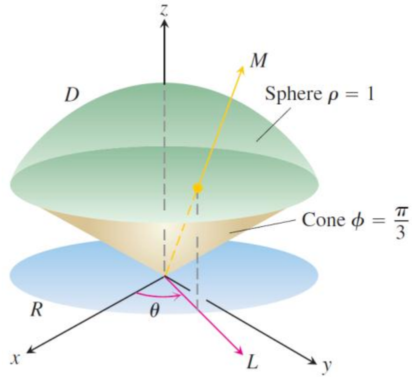
CYLINDRICAL TO RECTANGULAR	SPHERICAL TO CYLINDRICAL	SPHERICAL TO RECTANGULAR
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

Volume: $dV = dx\,dy\,dz \quad = dz\,r\,dr\,d\theta \quad = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

EXAMPLE 16: Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$. then find moment of inertia about the z-axis.

Solution:
$0 \le \rho \le 1\,,\ 0 \le \phi \le \pi/3\,,\ \ 0 \le \theta \le 2\pi$



$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi)$$

In rectangular coordinates, the moment is

$$I_z = \iiint (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi$. Hence,

$$I_z = \iiint (\rho^2\sin^2\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \iiint \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta.$$

$$I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi\, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\left(-\frac{1}{2}+1+\frac{1}{24}-\frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi}\frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.$$