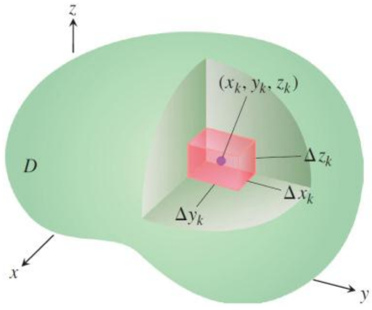
Triple Integrals in Rectangular Coordinates

triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three dimensional region.

Triple Integrals



If F(x, y, z) is a function defined on a closed bounded region D in space.

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k.$$

$$\boxed{\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\, dV \qquad \text{or} \qquad \lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz.}$$

We call this limit the **triple integral of *F* over *D*** and write

Volume of a Region in Space.

The volume of a closed, bounded region D in space is

$$\boxed{V = \iiint_D dV.}$$

Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iiint_D kF\, dV = k\iiint_D F\, dV$ (any number k)

2. *Sum and Difference:* $\displaystyle\iiint_D (F \pm G)\, dV = \iiint_D F\, dV \pm \iiint_D G\, dV$

3. *Domination:*

   (a) $\displaystyle\iiint_D F\, dV \ge 0$ if $F \ge 0$ on D

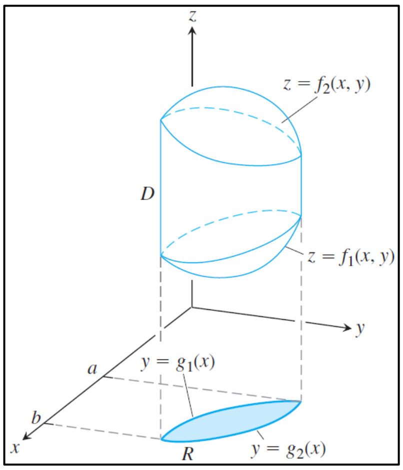
   (b) $\displaystyle\iiint_D F\, dV \ge \iiint_D G\, dV$ if $F \ge G$ on D

4. *Additivity:* $\displaystyle\iiint_D F\, dV = \iiint_{D_1} E\, dV + \iiint_{D_2} F\, dV$

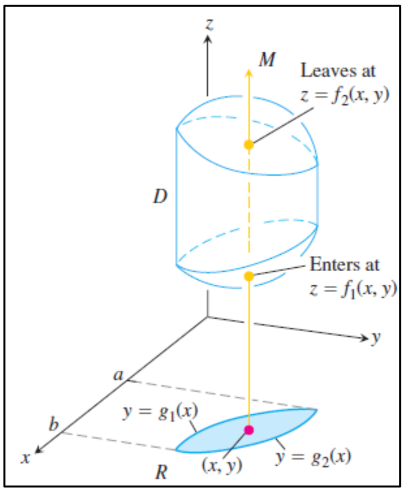
   if D is the union of two nonoverlapping regions D_1 and D_2.

Finding Limits of Integration

1. ***Sketch***: Sketch the region *D* along with its "shadow" *R* (vertical projection) in the *xy*-plane. Label the upper and lower bounding surfaces of *D* and the upper and lower bounding curves of *R*.

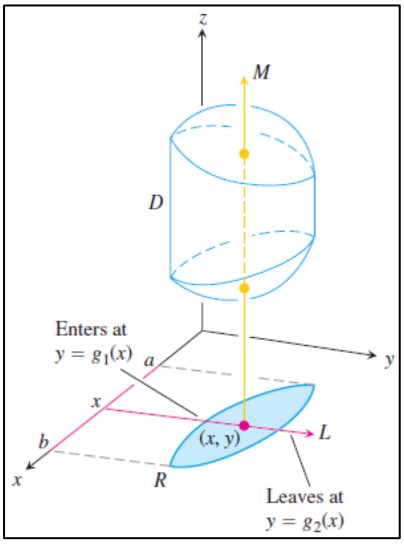


2. ***Find the z-limits of integration***: Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$ These are the z-limits of integration.



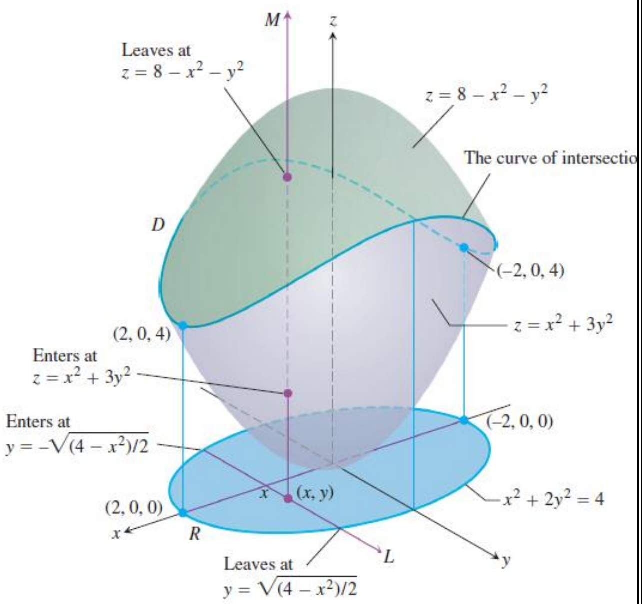
3. ***Find the y-limits of integration***: Draw a line L through *(x, y)* parallel to the *y-axis*. As y increases, *L* enters *R* ,at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the *y*-limits of integration.

4. ***Find the x-limits of integration***: Choose *x*-limits that include all lines through *R* parallel to the *y*-axis ($x = a$ and $x = b$ in the preceding figure). These are the *x*-limits of integration. The integral is



$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

EXAMPLE 11: Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Solution:

1. We first sketch the region
2. find the z-limits of integration
   enters $z_1 = x^2 + 3y^2$, leaves $z_2 = 8 - x^2 - y^2$.
3. find the y-limits of integration

$x^2 + 3y^2 = 8 - x^2 - y^2$

➡ $[\, 2x^2 + 4y^2 = 8 \,]$ % 2

➡ $x^2 + 2y^2 = 4$ or $y = \pm\sqrt{\frac{4-x^2}{2}}$

enters $y = -\sqrt{\frac{4-x^2}{2}}$, leaves $y = +\sqrt{\frac{4-x^2}{2}}$

4. Finally we find the x-limits of integration
   At $y = 0$ ➡ $x = \pm 2$

$$V = \iiint_D dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}(8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right) dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}.$$

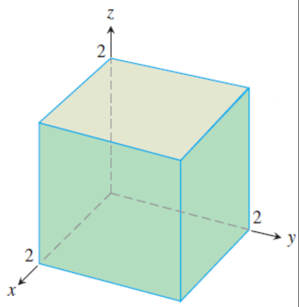
Let $x = 2\sin\theta$ $dx = 2\cos\theta\, d\theta$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

Average value of F over D = $\boxed{\dfrac{1}{\text{volume of } D}\iiint\limits_D F\, dV.}$

EXAMPLE 12: Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$ and $z = 2$ in the first octant.



Solution:

- ketch the cube with enough detail to show the limits of integration in the figure
- The volume of the cube is (2).(2).(2) = 8
- The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2\left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2\left[y^2z\right]_{y=0}^{y=2} dz = \int_0^2 4z\,dz = \left[2z^2\right]_0^2 = 8.$$

$$\begin{matrix}\text{Average value of}\\ xyz \text{ over the cube}\end{matrix} = \frac{1}{\text{volume}}\iiint\limits_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$