Vectors

3.1 Vectors and Scalars

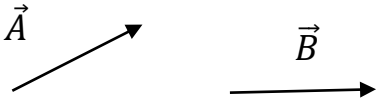
Vectors are mathematical quantities that are useful to describe physical quantities that have both a magnitude and direction associated with them.

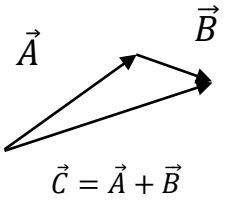
Vector: any quantity that has a magnitude and direction and behaves like the displacement, e.g. displacement, force, velocity, momentum, acceleration and torque.

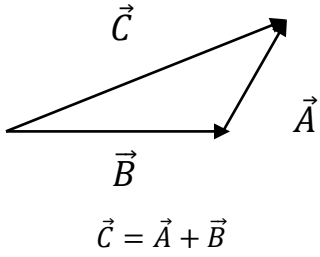
Scalar: any quantity that has a magnitude but no direction e.g. length, density, time, mass, temperature and energy.

3.2 Vector addition

To add two vector, $\vec{A}$ and $\vec{B}$, draw vector $\vec{A}$, then place the tail of $\vec{B}$ at the head of vector $\vec{A}$, then connect the tail of $\vec{A}$ to the head of $\vec{B}$ to obtain the vector (sum or resultant) $\vec{C}$:



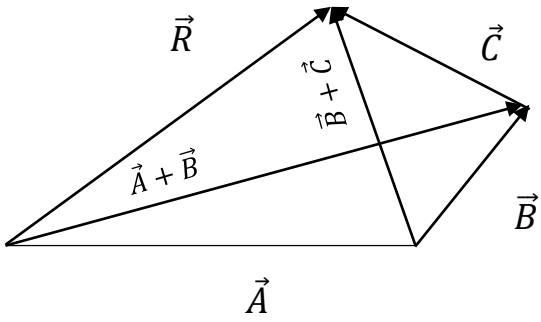




$= \vec{A} + \vec{B} = \vec{B} + \vec{A}$ (commutative law)

$= (\vec{A} + \vec{B}) + \vec{C}$

$= \vec{A} + (\vec{B} + \vec{A})$



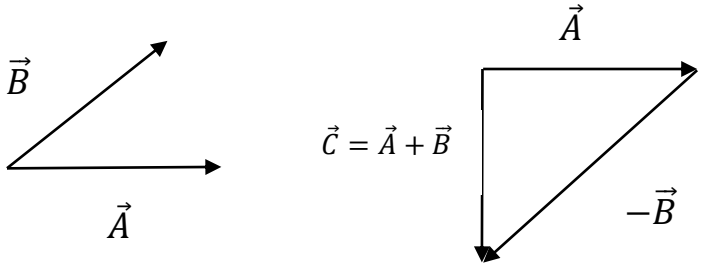
Let $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$

$\vec{A} - \vec{B} = (A_x - B_x)\hat{\imath} + (A_y - B_y)\hat{\jmath} + (A_z - B_z)\hat{k}$

Subtraction: The subtraction of two vectors $\vec{A}$ and $\vec{B}$ is defined as the sum of $\vec{A}$ and$(-\vec{B})$.

$\vec{C} = \vec{A} - \vec{B} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



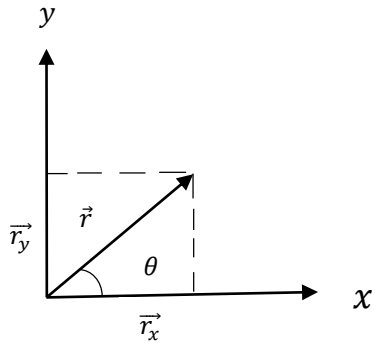
3.3 Vector Components

A vector is completely described by its components. Components can be used instead of vector itself. Components of vector is the projection of the vector on an axis

$\vec{r} = \vec{r}_x + \vec{r}_y$

$r_x = rcos\theta$

$r_y = rsin\theta$



The process of finding the components of a vector is called resolving the vector.

The magnitude direction of $\vec{r}$ are written as:

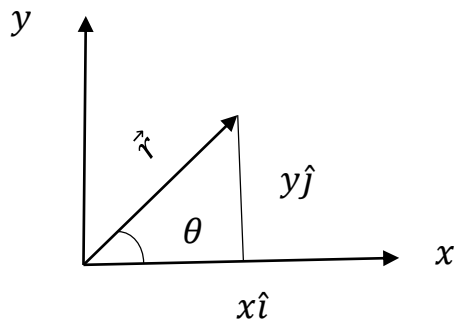
$r = \sqrt{r_x^2 + r_y^2}$

$tan\theta = \frac{r_y}{r_x}$

3.4 Unit Vector:

A unit vector is a vector that has a magnitude of exactly 1 and points in a particular direction. We shell use the symbols of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ to respect unit vector pointing in positive direction of x, y and z axis

For example, we can express $\vec{r}$ of figure as

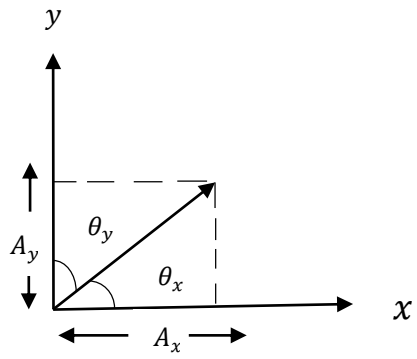


$\vec{r} = x\hat{\imath} + y\hat{\jmath}$

In 3-dimension (3-D)

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Example (1): 2-Dimension



$A_x = Acos\theta_x$

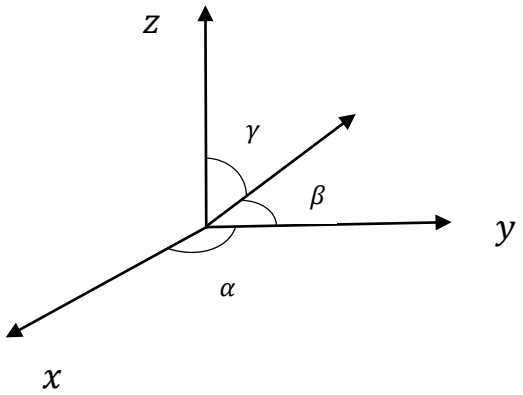
$A_y = Asin\theta_x$

magnitude of $\vec{A}$

$|A| = A = \sqrt{A_x^2 + A_y^2}$

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} = Acos\theta_x\hat{\imath} + Asin\theta_x\hat{\jmath}$

Example (2): General 3-D vector



$A_x = Acos\alpha$

$A_y = Acos\beta$

$A_z = Acos\gamma$

$cos\alpha^2 + cos\beta^2 + cos\gamma^2 = 1$

Magnitude of $\vec{A}$ vector is written as $\left|\vec{A}\right| = A,\ \left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$cos\alpha = \frac{A_x}{A}, etc.$

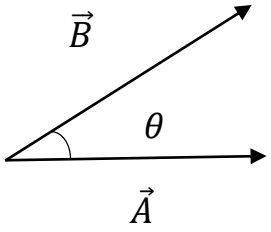
Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{A} = Acos\alpha\hat{\imath} + Acos\beta\hat{\jmath} + Acos\gamma\hat{k}$

3.5 Vector Multiplication

There are two ways to multiply vectors:

a. Dot product (scalar product): The scalar product of vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A}.\vec{B}$ and define to be:

$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\ cos\theta$ $\theta \leq 180^\circ$



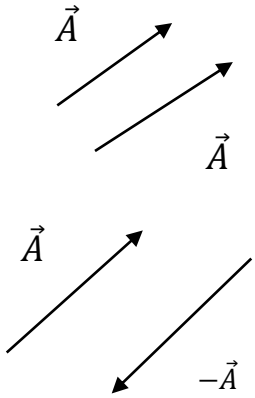
where $|\vec{A}|$ is the magnitude of $\vec{A}$.

$|\vec{B}|$ is the magnitude of $\vec{B}$.

θ is the angle between $\vec{A}$ and $\vec{B}$

Some properties of scalar product:

1. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (commutative)
2. $\vec{A}.\vec{A} = |\vec{A}||\vec{A}|cos\theta$
   $= A^2$ (Parallel product)
3. $\vec{A}.(-\vec{A}) = -A^2$ (anti parallel product)
4. $0 < \theta < 90^\circ$ $(\vec{A}.\vec{B}) > 0$
   $90^\circ < \theta < 180^\circ$ $(\vec{A}.\vec{B}) < 0$
   $\theta = 90^\circ$ $(\vec{A}.\vec{B}) = 0$
5. $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ (parallel unit vector)



$(\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{\imath} = 0)$,$(\hat{\imath}.\hat{k} = \hat{k}.\hat{\imath} = 0)$, $(\hat{\jmath}.\hat{k} = \hat{k}.\hat{\jmath} = 0)$ (perpendicular unit vector)

6. $(\vec{C}+\vec{D}).\vec{E}=\vec{C}.\vec{E}+\vec{D}.\vec{E}$ (distribution law)

$\vec{A}.\vec{B}=(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}).(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k})$

$[(A_xB_x\hat{\imath}.\hat{\imath}+A_xB_y\hat{\imath}.\hat{\jmath}+A_xB_z\hat{\imath}.\hat{k})+(A_yB_x\hat{\jmath}.\hat{\imath}+A_yB_y\hat{\jmath}.\hat{\jmath}+A_yB_z\hat{\jmath}.\hat{k})+$
$(A_zB_x\hat{k}.\hat{\imath}+A_zB_y\hat{k}.\hat{\jmath}+A_zB_z\hat{k}.\hat{k})]$

$$\vec{A}.\vec{B}=A_xB_x+A_yB_y+A_zB_z$$

What is $\hat{\jmath}.\hat{A}=\hat{\jmath}.(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})$?

Solution: it's equal to A_y

First application of a scalar product will be ***the concept of work*** which given by

$W=\vec{F}.\vec{d}$

Example 3: What is the right angle between $\vec{A}=3\hat{\imath}+7\hat{k}$ and $\vec{B}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$?

Solution:

$A_x=3$, $B_x=-1$

$A_y=0$, $B_y=2$

$A_z=7$, $B_z=1$

$\vec{A}.\vec{B}=A_xB_x+A_yB_y+A_zB_z$

$=(3)(-1)+(0)(2)+(7)(1)$

$=+4$

$|\vec{A}|=\sqrt{A_x^2+A_y^2+A_z^2}$

$= \sqrt{3^2 + 0^2 + 7^2} \implies |A| = \sqrt{58}$

$\left|\vec{B}\right| = \sqrt{B_x^2 + B_y^2 + B_z^2}$

$= \sqrt{(-1)^2 + 2^2 + 1^2} \implies \left|\vec{B}\right| = \sqrt{6}$

$cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|} \implies cos\theta = \frac{4}{\sqrt{58}\sqrt{6}}$

$\theta = cos^{-1}\frac{4}{\sqrt{58}\sqrt{6}}$

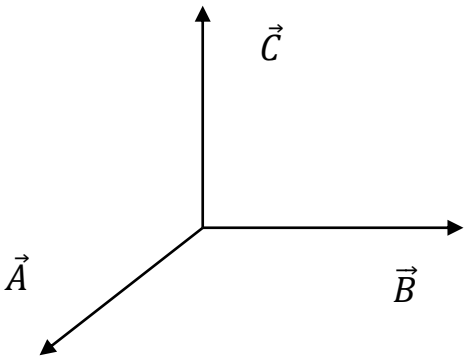
$\theta = 77.6^{\circ}$ [The angle between two vector $\vec{A}$ and $\vec{B}$]

b. Cross product (vector product):

The vector product of vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A} \times \vec{B}$ and define to be:

$\vec{A} \times \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| sin\theta \hat{n}$, where $\hat{n} = \frac{\vec{n}}{|n|}$

Where the vector product produced another vector perpendicular the plane that contain $\vec{A}$ and $\vec{B}$.



$\vec{C} \perp$ to both $\vec{A}$ and $\vec{B}$.

Some properties of vector product:

1. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
2. $\vec{A} \times \vec{A} = \left|\vec{A}\right|\left|\vec{A}\right| sin0 = 0$ [parallel vector]
3. $(\hat{\imath} \times \hat{\imath}) = (\hat{\jmath} \times \hat{\jmath}) = \left(\hat{k} \times \hat{k}\right) = 0$ [parallel unit vector]
4. $\left.\begin{matrix} \hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k} \\ \hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath} \\ \hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath} \end{matrix}\right\} Right\ hand\ coordinate\ system$

$\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$

$= (A_yB_z - A_zB_y)\hat{\imath} - (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$ [Result following multiplication term-by term and dimension of many vanishing terms]

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

3.6 Determinates

a. Order-2

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Example (4): $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (-2)(4) = 23$

b. Order-3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example (5): $\begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & 3 \\ -2 & 7 & 1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 3 \\ 7 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ -2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ -2 & 7 \end{vmatrix}$

$= 3[3(1) - 3(7)] - 2[4(1) - 3(-2)] + (-1)[4(7) - 3(-2)] = -108$

Example (6): if $\vec{A} = 3\hat{\imath} + 7\hat{\jmath} - \hat{k}$, $\vec{B} = \hat{\imath} - \hat{\jmath}$ what is $\vec{C} = \vec{A} \times \vec{B}$, $\vec{C}.\vec{A}$ and $\vec{C}.\vec{B}$?

Solution: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 7 & -1 \\ 1 & -1 & 0 \end{vmatrix}$

$= \hat{\imath}[7(0) - (-1)(-1)] - \hat{\jmath}[3(0) - (-1)(1)] + \hat{k}[3(-1) - 7(1)]$

$$= -\hat{\imath} - \hat{\jmath} + 10\hat{k}$$

$$\vec{C}.\vec{A} = \left(-\hat{\imath} - \hat{\jmath} + 10\hat{k}\right).\left(3\hat{\imath} + 7\hat{\jmath} - \hat{k}\right) = -3 - 7 + 10 = 0$$

$$\vec{C}.\vec{B} = \left(-\hat{\imath} - \hat{\jmath} + 10\hat{k}\right).(\hat{\imath} - \hat{\jmath}) = -1 + 1 = 0$$

$\vec{C} \perp$ to both $\vec{A}$ and $\vec{B}$

Example (7): find the magnitude and direction of the sum of three vector $\vec{A}, \vec{B}$ and $\vec{C}$ lying in the xy plane and given by:

$$\vec{A} = 4.2\hat{\imath} - 1.5\hat{\jmath}$$

$$\vec{B} = -1.6\hat{\imath} + 2.9\hat{\jmath}$$

$$\vec{C} = -3.7\hat{\jmath}$$

Solution:

$$R_x = A_x + B_x + C_x$$

$$4.2 - 1.6 + 0 = 2.6$$

$$R_y = A_y + B_y + C_y$$

$$= -1.5 + 2.9 - 3.7 = -2.3$$

$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath}$$

$$2.6\hat{\imath} - 2.3\hat{\jmath}$$

The magnitude of $\vec{R}$:

$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{(2.6)^2 + (-2.3)^2}$$

$R = 3.5$

The direction of $\vec{R}$:

$$tan\emptyset = \frac{R_y}{R_x} \Longrightarrow \emptyset = -41$$

3.7 Differentiation of Vector:

Let $\vec{A}(t)$ is a vector as a function to the variable t, where:

$$\vec{A}(t) = \hat{\imath}A_x(t) + \hat{\jmath}A_y(t) + \hat{k}A_z(t)$$

Then the differentiation process due to t is:

$$\frac{d\vec{A}}{dt} = \hat{\imath}\frac{dA_x}{dt} + \hat{\jmath}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt}$$

For two vector $\vec{A}$ and $\vec{B}$, we can write:

$$\frac{d}{dt}\left(\vec{A} + \vec{B}\right) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}\left(\vec{A}.\vec{B}\right) = \vec{B}.\frac{d\vec{A}}{dt} + \vec{A}.\frac{d\vec{B}}{dt}$$

Problems and Questions

Q.1: For vector $\vec{A} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{B} = +\hat{\imath} - \hat{\jmath} - 4\hat{k}$ find

1. $\vec{A} + \vec{B}$
2. $\vec{A} - \vec{B}$
3. $\vec{A}$ vector $\vec{C}$ such that $\vec{A} - \vec{B} + \vec{C} = 0$

Answer:

1. $3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$

2. $5\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$
3. Negative of 2.

Q.2: Show for any vector $\vec{A}$ that:

1. $\vec{A}.\vec{A} = A^2$
2. $\vec{A} \times \vec{A} = 0$

Q.3: Find the angle between two vectors given by:

$\vec{A} = 3\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$ and $\vec{B} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$

Q.4: a particle undergoes three consecutive displacement $d_1 = 15\hat{\imath} + 30\hat{\jmath} + 12\hat{k}$, $d_2 = 23\hat{\imath} - 14\hat{\jmath} - 5\hat{k}$ and $d_3 = -13\hat{\imath} + 15\hat{\jmath}$, find the components of the resultant displacement and its magnitude?

Q.5: Given the displacement vector $\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$ and $\vec{B} = 2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$, find the magnitude of the vector:

1. $\vec{C} = \vec{A} + \vec{B}$
2. $\vec{D} = 2\vec{A} - \vec{B}$
3. $\vec{E} = 2\vec{A} - (\vec{A} + \vec{B})$
4. $\vec{R} = \vec{A} \times \vec{B}$