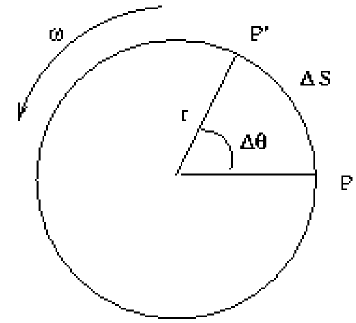
8.4 Rotation between Angular and linear velocity and acceleration

Consider a point p, p moves in a circle, the linear velocity vector is thus tangent to this circle. Magnitude is $\frac{ds}{dt}$, where s is distance travelled along the circular path



$s = r\theta \qquad [\theta \ in \ radian]$

$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$

$v = rw$

We can find the radiation between the linear acceleration and the angular acceleration, by taking the time derivative of v

$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$ [a_t the tangential acceleration]

$a_r = \frac{v^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2$ [a_r the radial acceleration]

Total linear acceleration of the particle is $\vec{a}$

$\vec{a} = \vec{a}_t + \vec{a}_r = r\alpha + r\omega^2 \qquad \left(\frac{m}{s^2}\right)$

8.5 Rotational Kinetic Energy

The kinetic energy K of a rigid body rotating about a fixed axis is given by:

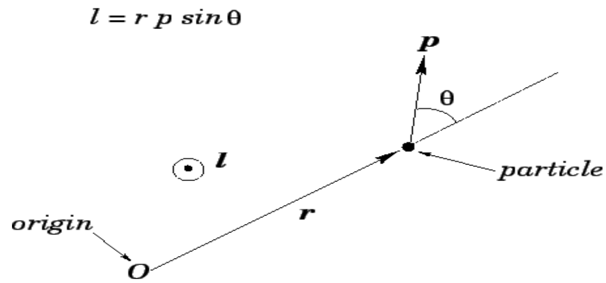
$K = \frac{1}{2}I\omega^2$

$I = \sum_i m_i r_i^2$ [moment of Inertia] $[I] = Kg.m^2$

ω, I ← Resistance to rotational motion

V, m ← Resistance to linear motion

8.6 Angular momentum of a Particle



Consider a particle of mass m, located at the position vector $\vec{r}$ and moving the velocity $\vec{v}$. The instantaneous angular momentum $\vec{L}$ of the particle relative to the origin "O" is defined by the cross product of its instantaneous position vector and it's instantaneous $\vec{p}$:

$$\vec{L} = \vec{r} \times \vec{p} = rp\ sin\theta \qquad \left[\frac{Kg.m}{s^2}\right]$$

By differentiating with respect to time

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \frac{d\vec{p}}{dt}$$

$\because \vec{v} \times \vec{v} = 0$, $\vec{F} = \frac{d\vec{p}}{dt}$ (Newton's law)

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

The quantity $\vec{r} \times \vec{F}$ is called a Torque (τ)

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad [N.m]$$

The rate of change of angular momentum equals the torque. The rotational analog of Newton's law is:

$$\tau = I\alpha$$

The torque acting on the particle is proportional to the angular acceleration

$$\vec{L} = constant \ \text{ if } \ \vec{\tau} \equiv 0$$

8.7 Work and Energy in Rotational Motion

The work done by the force $\vec{F}$ as the body rotates through a small distance

$ds = rd\theta$ in a time dt is

$dW = \vec{F}.d\vec{s} = (Fcos\emptyset)rd\theta$

$Fsin\emptyset =$ tangential component of $\vec{F}$

$\tau_z = (Fcos\emptyset)r$

$dW = \tau_z d\theta$

The power $P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z\omega$

8.8 Work-Energy Theorem in Rotational Motion

The work kinetic energy theorem for rotating bodies is:

$W = \Delta K = K_f - K_i = \frac{1}{2}IW_2^2 - \frac{1}{2}IW_1^2$

If the force acting is conservative, then the work done is:

$\frac{1}{2}IW_1^2 + U_1 + \frac{1}{2}IW_2^2 + U_2$ Conservation of mechanical energy in rotational motion or,

$E = \frac{1}{2}IW^2 + U = constant$

Example (1): A wheel rotates a constant angular acceleration of $3.5\left(\frac{m}{s^2}\right)$. If the angular speed of the wheel is $2\left(\frac{rad}{s}\right)$ at $t_i = 0$

a. What the angle does the wheel rotates in 2 s.

b. What is the angular speed at 2 s

c. Find the angle through which the wheel rotates between $t = 2s$ and $t = 3s$

Solution:

a. $\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = 2 \times 2 + \frac{1}{2} \times 3.5 \times (2)^2 = 11\, rad$

b. $\omega = \omega_0 + \alpha t = 2 + 3.5 \times 2 \left(\frac{rad}{sec}\right)$

c. $\Delta\theta = \theta_2 - \theta_1$

$at\ t = 2s \quad \theta_2 = 11\, rad$

$at\ t = 3s \ \Longrightarrow\ \theta_3 = \omega_0 t + \frac{1}{2}\alpha t^2 = 2 \times 3 + \frac{1}{2} \times 3.5(3)^2 = 21.75$

$\Delta\theta = \theta_2 - \theta_1 = 21.75 - 11 = 10.75\, rad$

Example (2): Record player rotates at 33 $\left(\frac{rev}{min}\right)$ and takes $20s$ to come the rest

a. What is the angular acceleration, assuming it is uniform.

b. How many rotations before it comes to rest.

c. If rim is the radius $r = 14cm,$ what is the acceleration of a point on the rim at $t = 0$

Solution:

a. $\omega_0 = 33 \left(\frac{rev}{min}\right)\left(2\pi \frac{rad}{rev}\right) \times \left(\frac{1min}{69s}\right) = 3.46 \left(\frac{rad}{s}\right)$

$\omega = \omega_0 + \alpha t$

$\omega = 0\ at\ t = 20s$

$\alpha = \frac{\omega - \omega_0}{t} = \frac{0-3.46}{20} = -0.173 \left(\frac{rad}{s^2}\right)$

b. $\Delta\theta = \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = 3.46(20) - \frac{1}{2}(0173)(20)^2 = 34.6rad$

c. $a_t = r\alpha = 14\left[0.173\left(\frac{rad}{s^2}\right)\right] = 2.42\left(\frac{cm}{s^2}\right)$

$a_c = r\omega^2 = 14(cm)(3.46)^2 = 168\left(\frac{cm}{s^2}\right)$

$a = \sqrt{2.42^2 + 168^2} = 168\left(\frac{cm}{s^2}\right)$

Velocity at rim $t = 0$

$$v = r\omega_0 = 14 \times 3.46 = 48.4\left(\frac{cm}{s}\right)$$

Example (3): A particle at rest($t = 0, \theta = 0, \omega = 0$), accelerated with a circular motion of radius 1.3m by the equation $\alpha = (120t^2 - 48t + 16)$ find

1. The angular position and the angular velocity as a function of t
2. The tangential and the normal acceleration components.

Solution: $\alpha = \frac{d\omega}{dt} = 120t^2 - 48t + 16$

$\int_0^\omega d\omega = 120\int_0^t t^2 dt - 48\int_0^t t dt + 16\int_0^t dt$

$\omega = 40t^3 - 24t^2 + 16t$

$\omega = \frac{d\theta}{dt} = 40t^3 - 24t^2 + 16t$

$\int_0^\theta d\theta = 40\int_0^t t^3 dt - 24\int_0^t t^2 dt + 16\int_0^t t\, dt$

$\theta = 10t^4 - 8t^3 + 8t^2$

$a_t = \alpha r = 1.3(120t^2 - 48t + 16)$

$a_r = r\omega^2 = 1.3(40t^3 - 24t^2 + 16t)^2$

Problems and Questions

Q.1: A disk start from rest at time $t = 0$, what is it rotational kinetic energy at

$t = 2.5s$?

Q.2: Uniform sphere, axis through center divide the sphere in to thin disks, calculate its momentum of inertia?

Q.3: A uniform solid cylinder has a radius R, mass M and length L, calculate the momentum of inertia about its central axis (the z-axis)

Q.4: Calculate the momentum of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y-axis) and passing through the center of mass