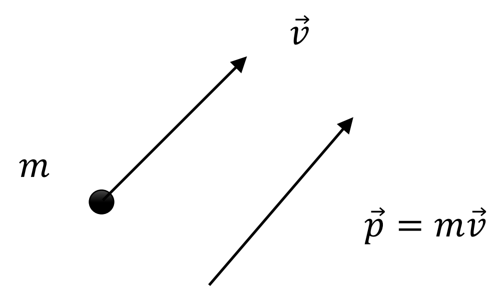
Center of Mass

7.1 System of Particles

Up to now we have studied motion of a single particle. We now want to look at a system of particles to see what we can learn about the motion. The momentum for a single particle $\vec{p} = m\vec{v}$, total momentum for a system of particle is the sum of the individual momentum

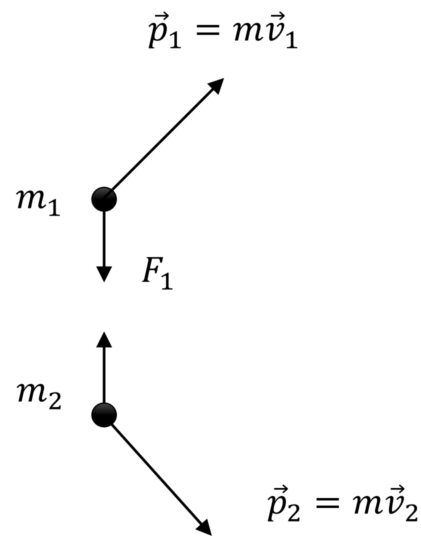


$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n = \sum_{i=1}^{n} p_i$$

Where $\vec{p}_1 = m\vec{v}_1$, $\vec{p}_2 = m\vec{v}_2$, ……..etc.

The simplest many – particle system in 2-particle system in which particles exert forces on each other by newton 3rd law

$$\vec{F}_1 = -\vec{F}_2$$



Equation of motion

$$\vec{F}_1 = \frac{d\vec{p}_1}{dt}, \quad \vec{F}_2 = \frac{d\vec{p}_2}{dt}$$

$$\vec{F}_1 + \vec{F}_2 = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

$$\therefore \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

$$\vec{p} = \vec{p}_1 + \vec{p}_1$$

- Particles exchange momentum as they interact
- If only internal forces act the total linear momentum is conserved

Consider 2-particle with external forces acting on them. Then

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{external}$$

Where $\vec{F}_{external}$ is total external force on system, if $\vec{F}_{external} = 0$

$$\frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = [constant]$$

7.2 Center of Mass

Up to now we have ignored the size of objects, we will now show that for an object of finite size. The position of the center of mass is the average position of the mass of the system.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n n}{m_1 + m_2 + \cdots + m_n}$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n r_n}{M} = \frac{\sum m_1\vec{r}_i}{M}$$

In terms of vector components:

$$x_{cm} = \frac{1}{M}[m_1x_1 + m_2x_2 + \cdots + m_nx_n] = \frac{1}{M}\sqrt{m_ix_i}$$

$$y_{cm} = \frac{1}{M}[m_1y_1 + m_2y_2 + \cdots + m_ny_n] = \frac{1}{M}\sqrt{m_iy_i}$$

$$z_{cm} = \frac{1}{M}[m_1z_1 + m_2z_2 + \cdots + m_nz_n] = \frac{1}{M}\sqrt{m_iz_i}$$

$$\vec{r}_{cm} = x_{cm}\hat{\imath} + y_{cm}\hat{\jmath} + z_{cm}\hat{k}$$

7.3 Motion of the Center of Mass

Suppose we take the time derivative of the position vector of the center of mass. Assuming m is constant (no particles center or leave system). Then we get the velocity of the center of mass.

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left[m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt}\right]$$

$$= \frac{1}{M}[m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n] = \frac{\vec{p}}{m}$$

$\vec{p} = m\vec{v}_{cm}$

$\frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left[m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} + \cdots + m_n\frac{d^2\vec{r}_n}{dt^2}\right]$

Total momentum of the system is its total mass multiplied by the velocity of the center of mass. Differentiate again to get the acceleration of the center of mass

$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum m_i\frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum m_i\vec{a}_i$

$M\vec{a}_{cm} = \sum \vec{F}_i \equiv$ Force of particle

The net force of the system is due only to the external force

$\sum \vec{F}_{ext} = M\vec{a}_{cm} = \frac{d\vec{p}}{dt}$

If $\sum \vec{F}_{ext} = 0$

$\frac{d\vec{p}}{dt} = M\vec{a}_{cm} = 0 \Longrightarrow \vec{p} = M\vec{v}_{cm} = constant \quad at\ \sum \vec{F}_{ext} = 0$

The total K.E is the sum of the individual particle K.E's

$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots + \frac{1}{2}m_nv_n^2 = \sum m_iv_i^2$

If internal and external conservative forces act on a body, the system will also have potential energy. For a system of particles:

$U = (m_1z_1 + m_2z_2 + \cdots + m_nz_n)g = Mz_{cm}g$

$U \equiv$[Function of position of all the particles]

7.4 Collision / Impulse

When two objects collide, the force they exert on each other usually act only for a short time. Such forces are called impulsive forces during the collision the impulsive force produce a large change in the motion of the object while any other forces present produce only small changes usually neglected. During a time interval dt, the momentum change by:

$$d\vec{p} = \vec{F}dt$$

Integrating over the time of collision

$$\vec{p}_f - \vec{p}_i = \int_{p_i}^{p_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}\, dt$$

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\, dt \quad \text{[impulse]}$$

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{J} \Longrightarrow \Delta\vec{p} = \vec{J} = \vec{F}\Delta t$$

7.5 Collisions

We want to study the collision of objects how they move (velocities) after collision. In special cases momentum conservation is sufficient. In general it is not enough. Can categorize collisions in terms of two types:

1. Elastic Collision

- Interaction forces are conservative
- Total kinetic energy is the same before and after collision
- Momentum is conserved $K_i = K_f,\ \vec{p}_i = \vec{p}_f$

2. Inelastic collision

- Momentum is conserved
- Total kinetic energy after collision is less than before

$K_i \neq K_f$, $\vec{p}_i = \vec{p}_f$

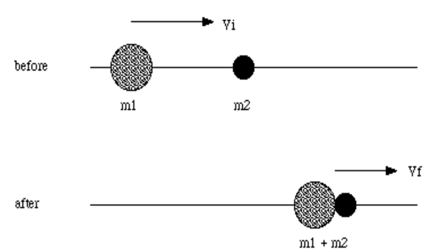
Inelastic Collision

Collision in which KE is not conserved are called inelastic collision. Some the energy is absorbed and converted to other forms. If the amount of KE absorbed is a maximum that is allowed by a momentum conservation, the collision said to be perfectly inelastic. In practice almost all collisions are inelastic to some degree.

1. Sticking collision- one dimension

Two particles masses m_1 and m_2 move with velocities v_{1i} and v_{2i} along a straight line. They collide stick, moving as a unit with velocity v_f after the collision.

Total momentum is conserved



$$\vec{p}_i = \vec{p}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i}+m_2\vec{v}_{2i}}{m_1+m_2}$$

2. Sticking collision- two dimension

- Linear momentum conserved
- Two equations, one for each component

$$tan\theta = \frac{m_2 v_2}{m_1 v_1}$$

$$v = \frac{m_2}{m_1+m_2}\frac{v_2}{sin\theta}$$

Elastic Collision

- Total energy is conserved
- Total linear momentum is conserved

1. Elastic Collision – One Dimension

- Assume a particles moving with velocities $\vec{v}_1$ and $\vec{v}_2$ before the collision
- Particles move with velocities $\vec{v}_1$ and $\vec{v}_2$ after the collision
- Conservation of momentum $\vec{p}_i = \vec{p}_f$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{\acute{v}}_1 + m_2\vec{\acute{v}}_2 \qquad (1)$$

- Conservation of energy $K_i = K_f$

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 \acute{v}_1^2 + \frac{1}{2}m_2 \acute{v}_2^2 \qquad (2)$$

Rewrite Eq. (1) $m_1\left(\vec{v}_1 - \vec{\acute{v}}_1\right) = m_2\left(\vec{\acute{v}}_2 - \vec{v}_2\right)$ (3)

Rewrite Eq. (2) $m_1(v_1^2 - \acute{v}_1^2) = m_2(v_2^2 - \acute{v}_2^2)$ (4)

$$\frac{(4)}{(3)} = \vec{v}_1 + \vec{\acute{v}}_1 = \vec{\acute{v}}_2 + \vec{v}_2 \qquad (5)$$

Or $\vec{v}_1 - \vec{v}_2 = \vec{\acute{v}}_2 - \vec{\acute{v}}_1$

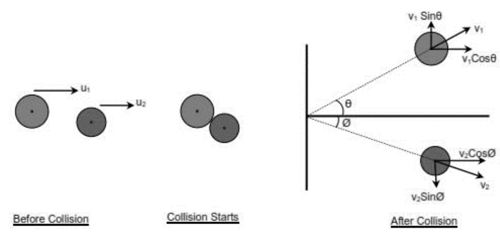
Now, if we substitute Eq. (1) in Eq. (5), we obtain

$$\acute{v}_2 = v_1\left(\frac{2m_1}{m_1+m_2}\right) + v_2\left(\frac{m_2-m_1}{m_1+m_2}\right)$$

$$\acute{v}_1 = v_1\left(\frac{m_1-m_2}{m_1+m_2}\right) + v_2\left(\frac{2m_1}{m_1+m_2}\right)$$

2. Elastic Collision – Two Dimension

When two bodies collide and their motion is not along single axis (the collision is not head on), the collision is two dimensional



- Conservational of momentum