

Masses and Moments in Three Dimensions(3-D)

This section shows how to calculate the masses and moments of three dimensional objects in Cartesian coordinates.

Mass and moment formulas for solid objects in space(3D)

If $\delta(x, y, z)$ is the density of material then

$$\text{Mass: } M = \iiint_D \delta \, dV \quad (\delta = \delta(x, y, z) = \text{density})$$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \delta \, dV, \quad M_{xz} = \iiint_D y \delta \, dV, \quad M_{xy} = \iiint_D z \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \iiint (y^2 + z^2) \delta \, dV$$

$$I_y = \iiint (x^2 + z^2) \delta \, dV$$

$$I_z = \iiint (x^2 + y^2) \delta \, dV$$

Moments of inertia about a line L :

$$I_L = \iiint r^2 \delta \, dV \quad (r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L)$$

Radius of gyration about a line L :

$$R_L = \sqrt{I_L/M}$$

EXAMPLE 13: Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

Solution:

By symmetry $\bar{x} = \bar{y} = 0$. To find \bar{z} , we first calculate

$$\begin{aligned} M_{xy} &= \iiint_R \int_{z=0}^{z=4-x^2-y^2} z \delta \, dz \, dy \, dx = \iint_R \left[\frac{z^2}{2} \right]_{z=0}^{z=4-x^2-y^2} \delta \, dy \, dx \\ &= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2 \, dy \, dx \\ &= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r \, dr \, d\theta \end{aligned}$$

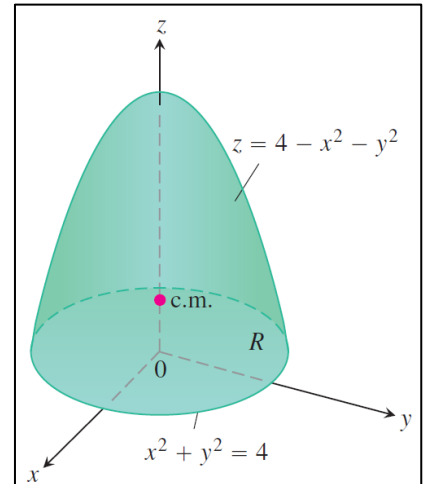
Polar coordinates

$$= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6} (4 - r^2)^3 \right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives

$$M = \iiint_R \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx = 8\pi\delta.$$

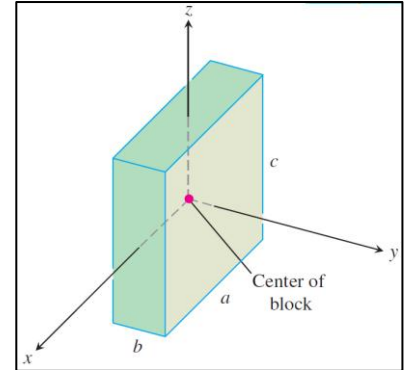
Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.



EXAMPLE 14: Find I_x, I_y, I_z for the rectangular solid of constant density shown in Figure .

Solution:

$$-\frac{a}{2} \leq x \leq \frac{a}{2}, \quad -\frac{b}{2} \leq y \leq \frac{b}{2}, \quad -\frac{c}{2} \leq z \leq \frac{c}{2}$$



$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2) \delta \, dx \, dy \, dz.$$

$$\begin{aligned} I_x &= 8 \int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2) \delta \, dx \, dy \, dz = 4a\delta \int_0^{c/2} \int_0^{b/2} (y^2 + z^2) \, dy \, dz \\ &= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y \right]_{y=0}^{y=b/2} dz \\ &= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2} \right) dz \\ &= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48} \right) = \frac{abc\delta}{12} (b^2 + c^2) = \frac{M}{12} (b^2 + c^2). \end{aligned}$$

Similarly,

$$I_y = \frac{M}{12} (a^2 + c^2) \quad \text{and} \quad I_z = \frac{M}{12} (a^2 + b^2).$$