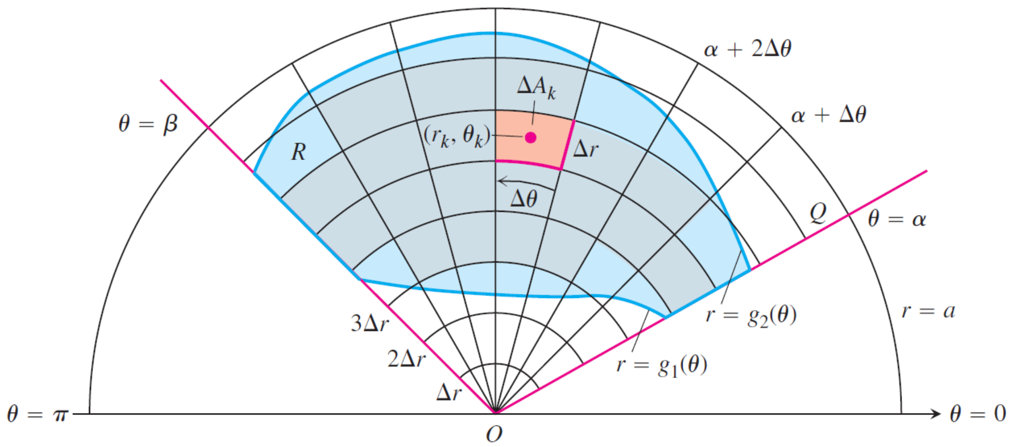
Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

- Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$.
- Suppose also that $0 \le g1(\theta) \le g2(\theta) \le a$ for every value of θ between α and β Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$.



Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \, \Delta r \, \Delta\theta.$$

As $n \to \infty$ and the values of Δr and $\Delta\theta$ approach zero, these sums converge to the double integral

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r \, dr \, d\theta.$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

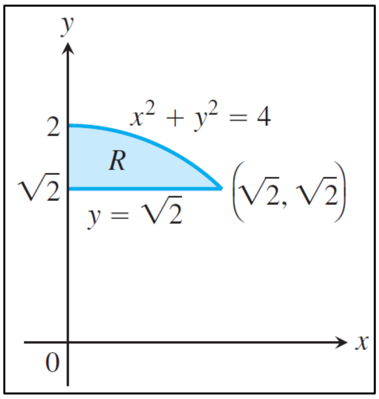
$$\boxed{\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r \, dr \, d\theta.}$$

Finding Limits of Integration

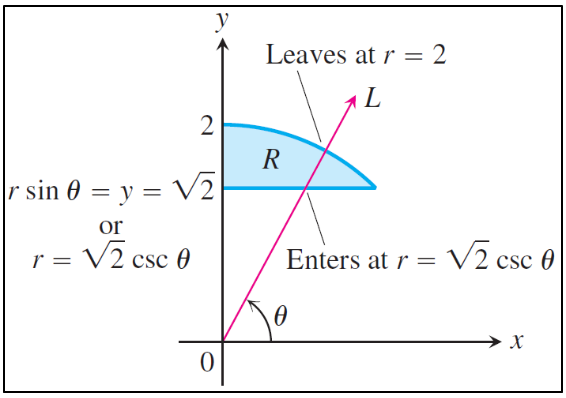
EXAMPLE: Find the limits of integration $f(r, \theta)$ for integrating over the region R that lies inside the circle $x^2+y^2=4$ and outside the $y = \sqrt{2}$.

Solution:

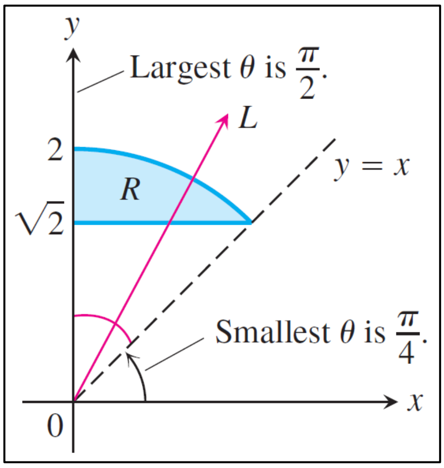
1. *Sketch:* Sketch the region and label the bounding curves.



2. *Find the r-limits of integration:* Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the θ angle that L makes with the positive x-axis.



3. *Find the θ-limits of integration*: Find the smallest and largest θ -values hat bound R. These are the u-limits θ *-limits*



The integral is

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r, \theta)\, r\, dr\, d\theta.$$

Area in Polar Coordinates

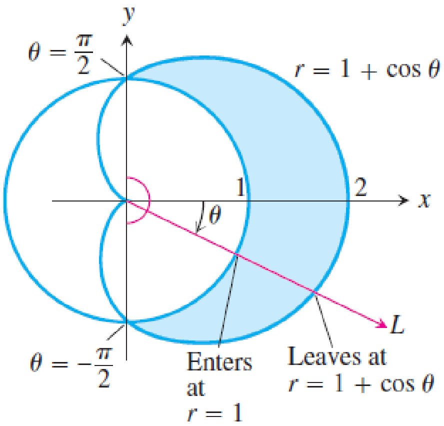
The area of a closed and bounded region R in the polar coordinate plane is

$$\boxed{A = \iint_R r\,dr\,d\theta.}$$

EXAMPLE 7: Find the limits of integration $f(r, \theta)$ for integrating over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution:

1. We first sketch the region and label the bounding curves.



2. Next we find the r-limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

3. Finally we find the u-limits of integration. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$ the integral is

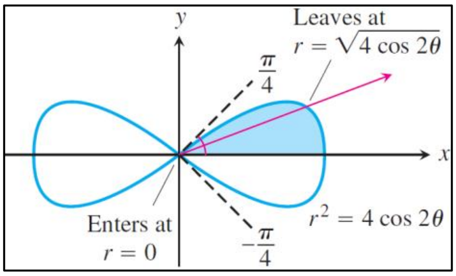
$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

EXAMPLE 8: Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution: Graph the lemniscate to determine the limits of integration and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



Changing Cartesian Integrals into Polar Integrals

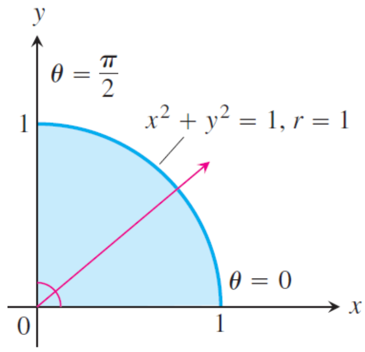
The procedure for changing a Cartesian integral $\iint_R f(x, y)\,dx\,dy$ by into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$ and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R. The Cartesian integral then becomes where G denotes R
The region of integration in polar coordinates.

$$\boxed{\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

EXAMPLE 9: Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution: We sketch the plate to determine the limits of integration . In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$



Integration with respect to y gives

$$\int_0^1 \left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. Substituting

$x = r\cos\theta, y = r\sin\theta$ and replacing $dx\, dy$ by $r\, dr\, d\theta$, we get

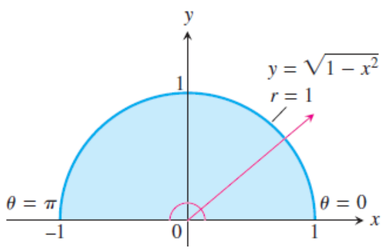
$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants.

EXAMPLE 10: Evaluating integrals using Polar coordinates $\iint_R e^{x^2+y^2}\, dy\, dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$



$0 \le r \le 1, \quad 0 \le \theta \le \pi.$

Solution:

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi} \int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi} \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$