Ministry of Higher Education and Scientifics research
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Three Stage
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Chapter Three
The Elementary Function

3.1 The exponential function:-

The exponential function defined as follow

$$f(Z) = e^{Z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$$

$$\therefore \boxed{f(Z) = e^{x}(\cos y + i\sin y)}$$

3.1.1 Characteristics:-

1) The absolute of exponential function is e^{x} ?

   Proof:-

   $|e^{Z}| = |e^{x+iy}| = |e^{x}e^{iy}| = |e^{x}||e^{iy}| = e^{x}|\cos y + i\sin y|$

   $= e^{x}(\cos^2 y + \sin^2 y)^{\frac{1}{2}} = e^{x}(1)^{\frac{1}{2}} = e^{x} \Longrightarrow \boxed{|e^{Z}| = e^{x}}$

2) The argument of exponential function is

   $arg(e^{Z}) = y + 2\pi k \qquad k = 0, \pm 1, \pm 2, \ldots..$

3) Polar form of exponential function is

   $\boxed{f(Z) = P(\cos\theta + i\sin\theta)} \qquad , where\ P = e^{x} = |e^{Z}|\ and\ \theta = y.$

4) $e^{Z} \neq 0$ Every where?

   Proof:-

   By polar form

   $e^{Z} = P(\cos\theta + i\sin\theta) = Pe^{i\theta}$

   Then $P = |e^{Z}| = e^{x} > 0$ & $e^{i\theta} = \cos\theta + i\sin\theta \qquad ,0 < \theta < 2\pi$

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So that $P > 0$ & $e^{i\theta} > 0$

Thus $e^{Z} \neq 0$

5)

a) $e^{0} = 1$

b) $e^{Z_1} e^{Z_2} = e^{Z_1+Z_2}$

c) $\frac{e^{Z_1}}{e^{Z_2}} = e^{Z_1-Z_2}$

d) $(e^{Z}) = e^{\bar{Z}}$

e) $e^{Z} = e^{Z+2\pi i}$

EX(1):- Find all value of Z such that $e^{Z} = 1 + i\sqrt{3}$?

Sol:-

$$e^{x} = |e^{Z}| = \left|1 + i\sqrt{3}\right| = \sqrt{1+3} = \sqrt{4} = 2$$

$$\therefore e^{x} = 2 \Longrightarrow x = log2$$

$$\tan y = \frac{\sqrt{3}}{1} \Longrightarrow y = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} + 2k\pi, k = 0, \pm 1, \pm 2, \ldots$$

$$\text{Thus } Z = x + iy = log2 + i\left(\frac{\pi}{3} + 2k\pi\right), k = 0, \pm 1, \pm 2, \ldots$$

EX(2):- Prove that $\left|e^{2Z+i} + e^{iZ^2}\right| \leq e^{2x} + e^{-2xy}$?

Sol:- $\left|e^{2Z+i} + e^{iZ^2}\right|^2 = \left|e^{2x+2yi+i} + e^{i[(x^2-y^2)+2xyi}\right|^2$

$$= \left|e^{2x+i(2y+1)} + e^{(x^2-y^2)i-2xy}\right|^2$$

$$= |e^{2x}[\cos(2y+1) + i\sin(2y+1)] + e^{-2xy}[\cos(x^2-y^2) + i\sin(x^2-y^2)]|^2$$

$$= |e^{2x}\cos(2y+1) + ie^{2x}\sin(2y+1) + e^{-2xy}\cos(x^2-y^2) + ie^{-2xy}\sin(x^2-y^2)|^2$$

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$$= |e^{2x}\cos(2y+1) + e^{-2xy}\cos(x^2-y^2) + i[e^{2x}\sin(2y+1) + e^{-2xy}\sin(x^2-y^2)]|^2$$

$$= (e^{2x}\cos(2y+1) + e^{-2xy}\cos(x^2-y^2))^2 + (e^{2x}\sin(2y+1) + e^{-2xy}\sin(x^2-y^2))^2$$

$$= e^{4x}\cos^2(2y+1) + 2e^{x}e^{-2xy}\cos(2y+1)\cos(x^2-y^2) + e^{-4xy}\cos^2(x^2-y^2) + e^{4x}\sin^2(2y+1) + 2\,e^{x}e^{-2xy}\sin(2y+1)\sin(x^2-y^2) + e^{-4xy}\sin^2(x^2-y^2)$$

$$= e^{4x}[\cos^2(2y+1) + \sin^2(\mathrm{y}+1)] + 2e^{x}e^{-2xy}[\cos(2y+1)\cos(x^2-y^2) + \sin(2y+1)\sin(x^2-y^2)] + e^{-4xy}[\cos^2(x^2-y^2) + \sin^2(x^2-y^2)$$

$$= e^{4x} + e^{-4xy} + 2e^{2x(1-y)}\cos[(2y+1)-(x^2-y^2)]$$

$$= e^{4x} + e^{-4xy} + 2e^{2x(1-y)}\cos(y^2+2y+1-x^2)$$

$$\le e^{4x} + e^{-4xy} + 2e^{2x(1-y)}$$

$$= (e^{2x})^2 + 2e^{2x}e^{-2xy} + (e^{-2xy})^2 = (e^{2x}+e^{-2xy})^2$$

$\left|e^{2Z+i} + e^{iZ^2}\right|^2 \le (e^{2x}+e^{-2xy})^2$ by root two sides, we get

$$\left|e^{2Z+i} + e^{iZ^2}\right| \le e^{2x} + e^{-2xy}$$