Ministry of Higher Education
and Scientifics research
University of Babylon

Complex Functions
Lecture 18

College of Education for Pure Sciences
Physics Department
Three Stage

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<u>EX(1)</u>:- Find a harmonic conjugate $v(x,y)$ of function $u(x,y) = e^x \cos y$ and $f(Z)$.

SOL: - by C.R. equation $u_x = v_y = e^x \cos y \Longrightarrow \frac{\partial v}{\partial y} = e^x \cos y$

$$\Longrightarrow \partial v = e^x \cos y \, \partial y \Longrightarrow \int \partial v = \int e^x \cos y \, \partial y$$

$$\Longrightarrow v = e^x \int \cos y \, \partial y + \emptyset\,(x) \;\Longrightarrow v = e^x \sin y + \emptyset(x) \;\text{———}\; *$$

$v_x = e^x \sin y + \emptyset'(x)$ But by C.R. $u_y = -v(x)$

$$\Longrightarrow u_y = -e^x \sin y \Longrightarrow v_x = e^x \sin y$$

$$e^x \sin y + \emptyset'\,(x) = e^x \sin y \Longrightarrow \emptyset'\,(x) = 0 \Longrightarrow \emptyset(x) = c$$

put in $*$ we get

$$v = e^x \sin y + c_1 \;\; thus \;\; f(Z) = e^x \cos y + i(e^x \sin y + c_1)$$

$$= e^x \cos y + ie^x \sin y + ic_1$$

$$f(Z) = e^x(\cos y + i \sin y) + c \qquad where \; c = ic_1$$

$$= e^x e^{iy} + c = e^{x+iy} + c = e^Z + c$$

<u>Theorem</u>:- The equation $u_{xx} + u_{yy} = 0$ is an equivalent equation $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ and $u_{rr} + v_{yy} = 0$ is an equivalent equation $r^2 v_{rr} + r v_r + u_{\theta\theta} = 0$.

<u>Proof</u>:-To prove $u_{xx} + u_{yy} = 0$ is equivalent

$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ By C-R equation

$r u_r = v_\theta \;\&\; u_\theta = -r v_r$

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We derivative first equation for r .

$$r u_{rr} + u_r(1) = v_{\theta r} \Longrightarrow r u_{rr} + u_r = v_{\theta r} \qquad \text{--------}*$$

We derivative second equation for θ .

$$u_{\theta\theta} = -r v_{r\theta} \Longrightarrow -\frac{1}{r} u_{\theta\theta} = v_{r\theta} \qquad \text{———} **$$

Since $v_{\theta r} = v_{r\theta}$, then

$$r u_{rr} + u_r = -\frac{1}{r} u_{\theta\theta} \Longrightarrow r u_{rr} + u_r - \frac{1}{r} u_{\theta\theta} = 0$$

To prove $v_{xx} + v_{yy} = 0$ is an equivalent

$r^2 v_{rr} + r v_r + v_{\theta\theta} = 0$ By C-R equation

$r u_r = v_\theta \quad \& \quad u_\theta = -r v_r$

We derivative first equation for θ .

$$r u_{r\theta} = v_{\theta\theta} \Longrightarrow u_{r\theta} = \frac{1}{2} v_{\theta\theta}$$

We derivative second equation for r .

$$u_{\theta r} = -r u_{rr} + v_r(-1) = -r u_{rr} - u_r \Longrightarrow u_{\theta v} - r u_{rr} - v_r \qquad \text{———} *$$

REMARK: - Since $Z = x + iy \quad \& \quad f(Z)$ then

$$f(Z) = u(x,y) + iv(x,y)$$

$$\frac{\partial f}{\partial x} = \frac{df}{dZ} \cdot \frac{\partial Z}{\partial x} = u_x(x,y) + i v_x(x,y)$$

$$\frac{df}{dZ}(1) = u_x(x,y) + i v_x(x,y)$$

Proof:- Since $Z = re^{i\theta} \quad \& \quad f(Z) = u(r,\theta) + iv(r,\theta)$

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Then

$$\frac{\partial f}{\partial r}=\frac{df}{dZ}\cdot\frac{\partial Z}{\partial r}=u_r+iv_r$$

$$\frac{\partial Z}{\partial r}=e^{i\theta}\quad \&\quad f'(Z)=\frac{df}{dZ}$$

$$f'(Z)=e^{i\theta}=u_r+iv_r\Longrightarrow f'(Z)=e^{-i\theta}(u_r+iv_r) \qquad *$$

$$\frac{\partial f}{\partial \theta}=\frac{df}{dZ}\cdot\frac{\partial Z}{\partial \theta}=u_\theta+iv_\theta$$

$$\frac{\partial Z}{\partial \theta}=ire^{i\theta}\quad \&\quad f'(Z)=\frac{df}{dZ}$$

$$f'(Z)ire^{i\theta}=u_\theta+iv_\theta\Longrightarrow f'(Z)=\frac{e^{-i\theta}}{ir}(u_\theta+iv_\theta) \qquad **$$

From $*$ and $**$ we get

$$\cancel{e^{-i\theta}}=(u_r+iv_r)=\cancel{e^{-i\theta}}\left(\frac{1}{ir}u_\theta+\frac{1}{r}v_\theta\right)$$

$$\Longrightarrow u_r+iv_r=\frac{-i}{r}u_\theta+\frac{1}{r}v_\theta\Longrightarrow u_r+iv_r=\frac{1}{r}v_\theta-i\frac{1}{r}u_\theta$$

$$\Longrightarrow u_r=\frac{1}{r}v_\theta\quad \&\quad u_r=-\frac{1}{r}u_\theta$$