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3.3 Trigonometric Function:-

From Euler formula, we get

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

Thus $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

If we suppose $Z = x + iy$, we get

$\sin Z = \frac{e^{iZ} - e^{-iZ}}{2i}$, $\cos Z = \frac{e^{iZ} + e^{-iZ}}{2}$

3.3.1 Characteristics:-

1) $\tan Z = \frac{\sin Z}{\cos Z}$, $\cos Z \neq 0$

$\cot Z = \frac{\cos Z}{\sin Z}$, $\sin Z \neq 0$

$\sec Z = \frac{1}{\cos Z}$, $\cos Z \neq 0$

$\csc Z = \frac{1}{\sin Z}$, $\sin Z \neq 0$

2) $\tan Z$, $\sec Z$ are analytic function in all point Zplan except $\cos Z = 0$.

Also $\cot Z$, $\csc Z$ are analytic functions in all point Zplans $\sin Z = 0$.

3) Differential

$\frac{d}{dZ}(\sin Z) = \cos Z$, $\frac{d}{dZ}(\cos Z) = -\sin Z$

$\frac{d}{dZ}(\tan Z) = \sec^2 Z$, $\frac{d}{dZ}(\cot Z) = -\csc^2 Z$

$\frac{d}{dZ}(\sec Z) = \sec Z \tan Z$, $\frac{d}{dZ}(\csc Z) = -\csc Z \cot Z$

4) Relations

1. $\sin^2 Z + \cos^2 Z = 1$
2. $\sin(-Z) = -\sin(Z)$
3. $\cos(-Z) = \cos Z$

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4. $\sin(Z_1 \pm Z_2) = \sin Z_1 \cos Z_2 \pm \sin Z_2 \cos Z_1$
5. $\cos(Z_1 \pm Z_2) = \cos Z_1 \cos Z_2 \mp \sin Z_1 \sin Z_2$
6. $\cos(2Z) = \cos^2 Z - \sin^2 Z$
7. $\sin(2Z) = 2 \sin Z \cos Z$

5) How make $\sin Z = u + iv$

$$\sin Z = \frac{1}{2i}\left(e^{iZ} - e^{-iZ}\right) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right) = \frac{1}{2i}\left(e^{-y}[\cos x + i \sin x] - e^{y}[\cos x - i \sin x]\right)$$

$$= \frac{1}{2i}\left(e^{-y} \cos x + i\, e^{-y} \sin x - e^{y} \cos x + i\, e^{y} \sin x\right)$$

$$= \frac{1}{2i}\left(\cos x \,(e^{-y} - e^{y}) + i \sin x \,(e^{-y} - e^{y})\right)$$

$$= \frac{1}{i} \cos x \frac{(e^{-y} - e^{y})}{2} + \sin x \frac{(e^{-y} + e^{y})}{2}$$

$$= -i \cos x \frac{(e^{-y} - e^{y})}{2} + \sin x \;\cosh y$$

$$= \sin x \;\cosh y + i \cos x \frac{(e^{y} - e^{-y})}{2}$$

$$\boxed{\sin Z = \sin x \;\cosh y + i \cos x \;\sinh y}$$

<u>H.W</u>:- $\cos z = \cos x \cosh y - i \sin x \sinh y$

<u>H.W</u>:- $|\sin Z|, |\cos Z|$

<u>EX:-</u>

1. $|\sin Z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2$

$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x\,(1 + \sinh^2 y) + (1 - \sin^2 x) \sinh^2 y$

$= \sin^2 x \cancel{+ \sin^2 x \sinh^2 y} + \sinh^2 y \cancel{- \sin^2 x \sinh^2 y}$

$= \sin^2 x + \sinh^2 y$

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2. $|\cos Z|^2 = \cos^2 x + \sinh^2 y$

<u>EX</u> (1):- Prove that

1. $\overline{\sin Z} = \sin \bar{Z}$

$\overline{\sin Z} = \overline{\sin x \cosh y + i \cos x \sinh y} = \sin x \cosh y - i \cos x \sinh y \ldots *$

$\sin \bar{Z} = \sin(x - iy) = \sin\big(x + i(-y)\big)$

$= \sin x \cosh(-y) + i \cos x \sinh(-y)$
$= \sin x \cosh y - i \cos x(-\sinh y)$
$= \sin x \cosh y - i \cos x \sinh y \ldots\ldots\ldots **$

2. $\overline{\cos Z} = \cos \bar{Z}$ **H.W**

<u>Ex (2)</u>:- Write the following function as $u + iv$?

1. $\cos i$
$x = 0 \;\; \& \;\; y = 1$
$\cos Z = \cos x \cosh y - i \sin x \sinh y$
$\cos i = \cos 0 \cosh 1 - i \sin 0 \sinh 1$
$= 1 \cosh 1 - i(0) \sinh 1 = \cosh 1$

2. $\sinh(\pi i)$ **H.W**

3. $\sec(1 - i)$ **H.W**

6) Since
$\sin Z = \sin x \cosh y + i \cos x \sinh y$
$\cos Z = \cos x \cosh y - i \sin x \sinh y$
If $x = 0$, then

i. $\sin iy = i \sinh y$
$\cos iy = \cosh y$ Put $t = iy$, then

ii. $\sin t = -i \sinh it$
$\cos t = \cosh it$