Ministry of Higher Education
and Scientifics research
University of Babylon

Complex Functions
Lecture 20

College of Education for Pure Sciences
Physics Department
Three Stage

Ali Hussein Mahmood Al-Obaidi
ali.alobaidi81@yahoo.com

3.2 <u>The logarithmic function</u>:-

The logarithmic function of Z is defined as follow

$\omega = \log Z = \log r + i\, org(Z)\, , Z \neq 0$

Where $r = |Z|$ & $org(Z) = \theta + 2\pi k \qquad k = 0, \pm 1, \pm 2, \dots..$

3.2.1 <u>Characteristics</u>:-

1) The logarithmic function is multiple valued function because any value k is unique branch of logarithmic function if $k = 0$,then the branch $\omega_0 = \log Z = \log r + i\theta \;,\; r > 0,\; 0 \leq \theta \leq 2\pi.$
is called principal value of logarithmic function or principal branch.

2) Any branch of logarithmic function is inverse function of exponential function e^Z **?**

Sol:- $e^{\log Z} = e^{\log|Z| + i\, arg(Z)} = e^{\log r + i(\theta + 2k\pi)}\; , k = 0, \pm 1, \pm 2$
$= e^{\log r} e^{i(\theta+2k\pi)} = r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]$
$= r(\cos\theta + i \sin\theta) = Z$

3) The principal value (branch) of logarithmic function is discontinuous in all points' non-negative real exist. So that it isn't analytic in this points because of the $\theta = 0$.

4) The principal value of logarithm function satisfy Cauchy-Riemann equations:-
SOL:-
$\log Z = u + iv = \log r + i\theta \Longrightarrow u = \log r \quad \& \quad v = \theta$
$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r} \;,\; \frac{\partial v}{\partial r} = 0 = \frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r}(0) = 0$

5) If $r > 0$ & $0 < \theta < 2\pi$ then the principle value of logarithmic function is analytic function, and differentiation of it is
$\omega_0 = \log Z = \log r + i\theta \;\; \& \;\; \omega_0 = u(r,\theta) + iv(r,\theta)$

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Where $u(r,\theta) = \log r$

$v(r,\theta) = \theta$

$$\frac{\partial \omega_0}{\partial r} = \frac{d\omega_0}{dZ} \cdot \frac{\partial Z}{\partial r}, Z = re^{i\theta}$$

But $\frac{d\omega_0}{dZ} = \frac{d}{dZ}(\log Z)$

$$\frac{\partial \omega_0}{\partial r} = \frac{d}{dZ}(\log Z) \cdot e^{i\theta} \qquad *$$

$$\frac{\partial \omega_0}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r} + i(0) = \frac{1}{r} \qquad *$$

From $*$ & $**$ we get

$$\frac{d}{dZ}(\log Z) \cdot e^{i\theta} = \frac{1}{r} \Longrightarrow \frac{d}{dZ}(\log Z) = e^{-i\theta}\frac{1}{r}$$

Thus $\frac{d}{dZ}(\log Z) = \frac{1}{re^{i\theta}} = \frac{1}{Z}$

6)

a) $\log Z \cdot w = \log Z + \log w$

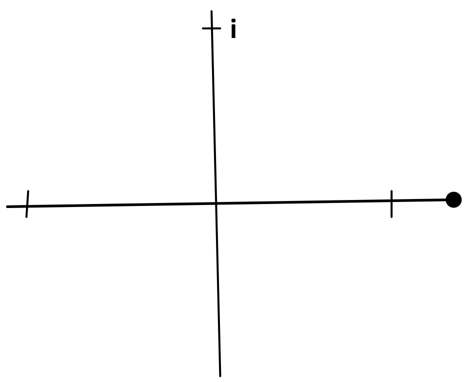
b) $\log \frac{Z}{w} = \log Z - \log w$

c) $\log e^Z = Z + 2k\pi i \qquad (k = 0, \pm1, \pm2, \dots.)$

d) $e^{\log Z} = Z$

e) $\log Z^{\frac{1}{k}} = \frac{1}{k}\log Z \qquad (k = \pm1, \pm2, \dots \dots.)$

<u>EX (1)</u>:-Prove that $\log(i)^2 \neq 2\log i$



SOL:-

$\log(i)^2 = \log(-1)$

$= \log|-1| + i(\pi + 2k\pi) \quad k = 0, \pm1, \pm2, \dots$

$= \log 1 + i\pi(1 + 2k) = \pi(1 + 2k)i$

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$$2\log i = 2\left((\log 1) + i\left(\frac{\pi}{2} + 2k\pi\right)\right)$$

$$= 2\left(\log 1 + i\pi\left(\frac{1}{2} + 2k\right)\right) = 2i\pi\left(\frac{1}{2} + 2k\right) = i\pi(1 + 4k)$$

EX:-Prove $\log(1+i)^2 = 2\log(1+i)$

SOL:-

$$\log(1+i)^2 = \log(1 + 2i - 1) = \log 2i = \log 2 + i\left(\frac{\pi}{2}\right)$$

$$2\log(1+i) = 2\left(\log\sqrt{2} + i\left(\frac{\pi}{4}\right)\right)$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$= 2\log\sqrt{2} + 2i\left(\frac{\pi}{4}\right) = \log\left(\sqrt{2}\right)^2 + i\frac{\pi}{2}$$

$$= \log 2 + i\left(\frac{\pi}{2}\right)$$

EX(3):- $\log(-1+i)^4 \neq 4\log(-1+i)$

Sol:-

$$\log(-1+i)^4 = \log((-1+i)^2)^2 = \log(1 - 2i - 1)^2 = \log -4$$

$$= \log 4 + i\pi$$

$$4\log(-1+i) = 4\left(\log\sqrt{2} + i\frac{3\pi}{4}\right) = \log 4 + 3\pi i$$

QUZE:-$\log(-1) = (2k+1)\pi i$

QUZE:-$\log(1-i)^2 \neq \log(1-i)$