

EX – Evaluate the following integrals:

$$1) \int \frac{\cosh(\ln x)}{x} dx$$

$$6) \int \operatorname{sech}^2(2x - 3) dx$$

$$2) \int \sinh(2x + 1) dx$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$3) \int \frac{\sinh x}{\cosh^4 x} dx$$

$$8) \int (e^{ax} - e^{-ax}) dx$$

$$4) \int x \cdot \cosh(3x^2) dx$$

$$9) \int \frac{\sinh x}{1 + \cosh x} dx$$

$$5) \int \sinh^4 x \cdot \cosh x dx$$

$$10) \int \operatorname{csch}^2 x \cdot \coth x dx$$

Sol.-

$$1) \int \cosh(\ln x) \cdot \left(\frac{dx}{x} \right) = \sinh(\ln x) + c$$

$$2) \frac{1}{2} \int \sinh(2x + 1) \cdot (2 dx) = \frac{1}{2} \cosh(2x + 1) + c$$

$$3) \int \frac{1}{\cosh^3 x} \cdot \frac{\sinh x}{\cosh x} dx = \int \operatorname{sech}^3 x \cdot \tanh x dx$$

$$= - \int \operatorname{sech}^2 x \cdot (-\operatorname{sech} x \cdot \tanh x dx) = -\frac{\operatorname{sech}^3 x}{3} + c$$

$$4) \frac{1}{6} \int \cosh(3x^2) \cdot (6x dx) = \frac{1}{6} \sinh(3x^2) + c$$

$$5) \int \sinh^4 x \cdot (\cosh x dx) = \frac{\sinh^5 x}{5} + c$$

$$6) \frac{1}{2} \int \operatorname{sech}^2(2x - 3) \cdot (2 dx) = \frac{1}{2} \tanh(2x - 3) + c$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \tanh x dx = \ln(\cosh x) + c$$

$$8) 2 \int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a} \int \sinh ax (a dx) = \frac{2}{a} \cosh ax + c$$

$$9) \int \frac{\sinh x dx}{1 + \cosh x} = \ln(1 + \cosh x) + c$$

$$10) - \int \operatorname{csch} x \cdot (-\operatorname{csch} x \cdot \coth x dx) = -\frac{\operatorname{csch}^2 x}{2} + c$$

Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$1) \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$2) \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$3) \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c$$

$$4) \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$5) \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

EX – Evaluate the following integrals:

$$1) \int \frac{dx}{\sqrt{1+4x^2}}$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$3) \int \frac{dx}{1-x^2}$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}}$$

$$5) \int \frac{\sec^2 \theta \, d\theta}{\sqrt{\tan^2 \theta - 1}}$$

$$6) \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})}$$

Sol.-

$$1) \frac{1}{2} \int \frac{2 \, dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$$

$$2) \int \frac{\frac{1}{2} \, dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1} \frac{x}{2} + c$$

$$3) \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x| < 1 \\ = \coth^{-1} x + c \quad \text{if } |x| > 1$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{\frac{1}{2} dx}{\frac{x}{2} \sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2} \operatorname{csc} h^{-1} \left| \frac{x}{2} \right| + c$$

$$5) \int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta d\theta) = \cosh^{-1}(\tan \theta) + c$$

$$6) \quad \text{let} \quad u = \ln \sqrt{x} = \frac{1}{2} \ln x \quad du = \frac{1}{2x} dx$$

$$\begin{aligned} \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1 - \ln^2 \sqrt{x})} &= \int \tanh^{-1} u \cdot \frac{2 du}{1 - u^2} \\ &= 2 \frac{(\tanh^{-1} u)^2}{2} + c = \left[\tanh^{-1}(\ln \sqrt{x}) \right]^2 + c \end{aligned}$$

Methods of integration

Integration by parts:

The formula for integration by parts comes from the product rule:-

$$d(u \cdot v) = u \cdot dv + v \cdot du \Rightarrow u \cdot dv = d(u \cdot v) - v \cdot du$$

and integrated to give: $\int u \, dv = \int d(u \cdot v) - \int v \, du$

then the integration by parts formula is:-

$$\int u \, dv = u \cdot v - \int v \, du$$

Rule for choosing u and dv is:

For u : choose something that becomes simpler when differentiated.

For dv : choose something whose integral is simple.

It is not always possible to follow this rule, but when we can.

EX – Evaluate the following integrals:

1) $\int x e^x \, dx$

6) $\int \ln(x + \sqrt{1+x^2}) \, dx$

2) $\int x \cdot \cos x \, dx$

7) $\int \sin^{-1} ax \, dx$

3) $\int \frac{x}{\sqrt{x-1}} \, dx$

8) $\int e^{ax} \cdot \sin bx \, dx$

4) $\int x^2 \cdot \ln x \, dx$

9) $\int x^3 \cdot e^x \, dx$

5) $\int x \cdot \sec^2 x \, dx$

10) $\int x^3 \cdot e^{x^2} \, dx$

Sol. –

1) let $\left. \begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} \Rightarrow \int u \, dv = u \cdot v - \int v \, du$

$$\int x \cdot e^x \, dx = x \cdot e^x - \int e^x \, dx = x \cdot e^x - e^x + c$$

$$2) \quad \left. \begin{array}{l} u = x \Rightarrow du = dx \\ dv = \cos x \, dx \Rightarrow v = \sin x \end{array} \right\} \Rightarrow \int u \, dv = u \cdot v - \int v \, du$$

$$\int x \cdot \cos x \, dx = x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x + \cos x + c$$

$$3) \quad \left. \begin{array}{l} u = x \Rightarrow du = dx \\ dv = \frac{1}{\sqrt{x-1}} \, dx \Rightarrow v = 2(x-1)^{1/2} \end{array} \right\} \Rightarrow \int u \, dv = u \cdot v - \int v \, du$$

$$\int \frac{x}{\sqrt{x-1}} \, dx = 2x \cdot (x-1)^{1/2} - 2 \int (x-1)^{1/2} \, dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3} \sqrt{(x-1)^3} + c$$

$$4) \quad \left. \begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x} \, dx \\ dv = x^2 \, dx \Rightarrow v = \frac{x^3}{3} \end{array} \right\} \Rightarrow \int u \, dv = u \cdot v - \int v \, du$$

$$\int x^2 \cdot \ln x \, dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{3} \int x^2 \, dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{9} x^3 + c$$

$$5) \quad \left. \begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sec^2 x \, dx \Rightarrow v = \tan x \end{array} \right\} \Rightarrow \int u \, dv = u \cdot v - \int v \, du$$

$$\int x \cdot \sec^2 x \, dx = x \cdot \tan x - \int \tan x \, dx = x \cdot \tan x + \ln |\cos x| + c$$

$$6) \quad \text{let } u = \ln(x + \sqrt{1+x^2}) \Rightarrow du = \frac{1 + \frac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \, dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \cdot \ln(x + \sqrt{1+x^2}) - \int x(1+x^2)^{-1/2} \, dx$$

$$= x \cdot \ln(x + \sqrt{1+x^2}) - \frac{1}{2} \cdot \frac{(1+x^2)^{1/2}}{1/2} + c = x \cdot \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + c$$

$$7) \quad \text{let } u = \sin^{-1} ax \Rightarrow du = \frac{a dx}{\sqrt{1-a^2 x^2}} \quad \& \quad dv = dx \Rightarrow v = x$$

$$\begin{aligned} \int \sin^{-1} ax dx &= x \cdot \sin^{-1} ax - \int \frac{a x}{\sqrt{1-a^2 x^2}} dx \\ &= x \cdot \sin^{-1} ax + \frac{1}{2a} \int -2a^2 x (1-a^2 x^2)^{-1/2} dx \\ &= x \cdot \sin^{-1} ax + \frac{1}{2a} \cdot \frac{(1-a^2 x^2)^{1/2}}{1/2} + c = x \cdot \sin^{-1} ax + \frac{\sqrt{1-a^2 x^2}}{a} + c \end{aligned}$$

$$8) \quad \text{let } u = e^{ax} \Rightarrow du = a \cdot e^{ax} dx \quad \& \quad dv = \sin bx dx \Rightarrow v = -\frac{1}{b} \cos bx$$

$$\int e^{ax} \cdot \sin bx dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b} \int e^{ax} \cdot \cos bx dx \quad \dots\dots\dots(1)$$

$$\text{let } u = e^{ax} \Rightarrow du = a \cdot e^{ax} dx \quad \& \quad dv = \cos bx dx \Rightarrow v = \frac{1}{b} \sin bx$$

$$\int e^{ax} \cdot \cos bx dx = \frac{1}{b} e^{ax} \cdot \sin bx - \frac{a}{b} \int e^{ax} \cdot \sin bx dx \quad \dots\dots\dots(2)$$

sub. (2) in (1) \Rightarrow

$$\int e^{ax} \cdot \sin bx dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx dx - \frac{a^2}{b^2} \int e^{ax} \cdot \sin bx dx$$

$$\int e^{ax} \cdot \sin bx dx + \frac{a^2}{b^2} \int e^{ax} \cdot \sin bx dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx dx + c$$

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cdot \sin bx dx = \frac{e^{ax}}{b^2} (a \sin bx - b \cos bx) + c$$

$$\therefore \int e^{ax} \cdot \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

9) derivative of u integration of dv

$$\begin{array}{lcl} x^3 & \xrightarrow{+} & e^x \\ 3x^2 & \xrightarrow{-} & e^x \\ 6x & \xrightarrow{+} & e^x \\ 6 & \xrightarrow{-} & e^x \\ 0 & & e^x \end{array}$$

$$\begin{aligned} \therefore \int x^3 e^{ax} dx &= x^3 e^x - 3x^2 e^x \\ &\quad + 6x e^x - 6e^x + c \\ &= e^x (x^3 - 3x^2 + 6x - 6) + c \end{aligned}$$

$$10) \quad \text{let } u = x^2 \Rightarrow du = 2x dx \quad \& \quad dv = x \cdot e^{x^2} dx \Rightarrow v = \frac{1}{2} e^{x^2}$$

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} \int 2x \cdot e^{x^2} dx = \frac{1}{2} x^2 \cdot e^{x^2} - \frac{1}{2} e^{x^2} + c$$

Odd and even powers of sine and cosine:

To integrate an odd positive power of $\sin x$ (say $\sin^{2n+1} x$) we split off a factor of $\sin x$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int \sin^{2n+1} x \cdot dx = \int (1 - \cos^2 x)^n \cdot \sin x \, dx$$

and $\int \cos^{2n+1} x \cdot dx = \int (1 - \sin^2 x)^n \cdot \cos x \, dx$

EX - Evaluate:

$$1) \int \sin^3 x \, dx$$

$$2) \int \cos^5 x \, dx$$

Sol.

$$\begin{aligned} 1) \int \sin^3 x \, dx &= \int \sin^2 x \cdot \sin x \, dx = \int (1 - \cos^2 x) \cdot \sin x \, dx \\ &= \int \sin x \, dx + \int \cos^2 x \cdot (-\sin x) \, dx = -\cos x + \frac{1}{3} \cos^3 x + c \end{aligned}$$

$$\begin{aligned} 2) \int \cos^5 x \, dx &= \int \cos^4 x \cdot \cos x \, dx = \int (1 - \sin^2 x)^2 \cdot \cos x \, dx \\ &= \int \cos x \, dx - 2 \int \sin^2 x \cdot \cos x \, dx + \int \sin^4 x \cdot \cos x \, dx \\ &= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c \end{aligned}$$

To integrate an even positive power of sine (say $\sin^{2n} x$) we use the relations:-

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1 - \cos 2x}{2} \right)^n dx$$

$$\text{and } \int \cos^{2n} x \cdot dx = \int \left(\frac{1 + \cos 2x}{2} \right)^n dx$$

EX - Evaluate:

$$1) \int \cos^2 \theta d\theta$$

$$2) \int \sin^4 \theta d\theta$$

Sol.

$$\begin{aligned} 1) \int \cos^2 \theta d\theta &= \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2} \left[\int d\theta + \frac{1}{2} \int 2 \cos 2\theta d\theta \right] \\ &= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right] + c \end{aligned}$$

$$\begin{aligned} 2) \int \sin^4 \theta d\theta &= \int \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4} \left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta d\theta \right] \\ &= \frac{1}{4} \left[\theta - \sin 2\theta + \int \frac{1 + \cos 4\theta}{2} d\theta \right] = \frac{1}{4} \left[\theta - \sin 2\theta + \frac{1}{2} \left(\theta + \frac{1}{4} \sin 4\theta \right) \right] + c \\ &= \frac{3}{8} \theta - \frac{1}{4} \sin 2\theta + \frac{1}{32} \sin 4\theta + c \end{aligned}$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \, dx \quad , \quad \int \sin mx \cdot \cos nx \, dx \quad , \text{and} \quad \int \cos mx \cdot \cos nx \, dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$

$$\sin mx \cdot \cos nx = \frac{\sin(m-n)x + \sin(m+n)x}{2}$$

$$\cos mx \cdot \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

EX - Evaluate:

$$1) \int \sin 3x \cdot \cos 5x \, dx \quad 2) \int \cos x \cdot \cos 7x \, dx \quad 3) \int \sin x \cdot \sin 2x \, dx$$

Sol.-

$$\begin{aligned} 1) \int \sin 3x \cdot \cos 5x \, dx &= \frac{1}{2} \int (\sin(3x - 5x) + \sin(3x + 5x)) \, dx \\ &= \frac{1}{2} \left[-\frac{1}{2} \int \sin 2x (2dx) + \frac{1}{8} \int \sin 8x (8dx) \right] = \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + c \end{aligned}$$

$$2) \int \cos x \cdot \cos 7x \, dx = \frac{1}{2} \int (\cos(6x) + \cos(8x)) \, dx = \frac{1}{12} \sin 6x + \frac{1}{16} \sin 8x + c$$

$$3) \int \sin x \cdot \sin 2x \, dx = \frac{1}{2} \int (\cos x - \cos 3x) \, dx = \frac{1}{2} \sin x - \frac{1}{6} \sin 3x + c$$

Trigonometric substitutions:

Trigonometric substitutions enable us to replace the binomials $a^2 - u^2$, $a^2 + u^2$, and $u^2 - a^2$ by single square terms. We can use:-

$$u = a \sin \theta \quad \text{for } a^2 - u^2 = a^2 - a^2 \sin^2 \theta = a^2 (1 - \sin^2 \theta) = a^2 \cos^2 \theta$$

$$u = a \tan \theta \quad \text{for } a^2 + u^2 = a^2 + a^2 \tan^2 \theta = a^2 (1 + \tan^2 \theta) = a^2 \sec^2 \theta$$

$$u = a \sec \theta \quad \text{for } u^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta$$

EX - Evaluate the following integrals:

$$1) \int \frac{z^5 \, dz}{\sqrt{1+z^2}}$$

$$4) \int \frac{x^2}{\sqrt{9-x^2}} \, dx$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$5) \int \frac{dt}{\sqrt{25t^2-9}}$$

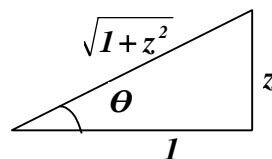
$$3) \int \frac{dx}{4-x^2}$$

$$6) \int \frac{dy}{\sqrt{25+9y^2}}$$

Sol.-

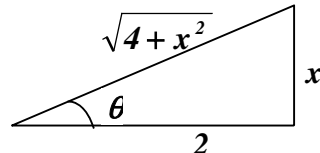
$$1) \text{ let } z = \tan \theta \Rightarrow dz = \sec^2 \theta \cdot d\theta \quad \tan \theta = \frac{z}{1}$$

$$\begin{aligned} \int \frac{z^5 dz}{\sqrt{1+z^2}} &= \int \frac{\tan^5 \theta \cdot \sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int \tan^5 \theta \cdot \sec \theta d\theta \\ &= \int \tan \theta \cdot \sec \theta (\sec^2 \theta - 1)^2 d\theta \\ &= \int \sec^4 \theta (\tan \theta \cdot \sec \theta d\theta) - 2 \int \sec^2 \theta (\tan \theta \cdot \sec \theta d\theta) + \int \tan \theta \cdot \sec \theta d\theta \\ &= \frac{1}{5} \sec^5 \theta - \frac{2}{3} \sec^3 \theta + \sec \theta + c \\ &= \frac{1}{5} (\sqrt{1+z^2})^5 - \frac{2}{3} (\sqrt{1+z^2})^3 + \sqrt{1+z^2} + c \end{aligned}$$



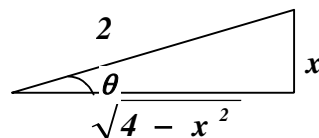
$$2) \text{ let } x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta \cdot d\theta \quad \tan \theta = \frac{x}{2}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4+4 \tan^2 \theta}} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + c \\ &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + c \\ &= \ln |\sqrt{4+x^2} + x| + c' \quad \text{where } c' = c - \ln 2 \end{aligned}$$



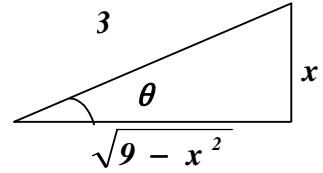
$$3) \text{ let } x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta \cdot d\theta$$

$$\begin{aligned} \int \frac{dx}{4-x^2} &= \int \frac{2 \cos \theta d\theta}{4-4 \sin^2 \theta} = \frac{1}{2} \int \frac{d\theta}{\cos \theta} = \frac{1}{2} \int \sec \theta d\theta \\ &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + c \\ &= \frac{1}{2} \ln \left| \frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}} \right| + c \\ &= \frac{1}{2} \ln \left| \frac{2+x}{\sqrt{(2-x)(2+x)}} \right| + c = \frac{1}{2} \ln \left| \sqrt{\frac{2+x}{2-x}} \right| + c = \frac{1}{4} \ln \left| \frac{2+x}{2-x} \right| + c \end{aligned}$$



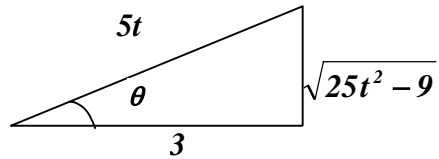
4) let $x = 3 \sin \theta \Rightarrow dx = 3 \cos \theta \cdot d\theta$

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9-x^2}} &= \int \frac{9 \sin^2 \theta}{\sqrt{9-9 \sin^2 \theta}} 3 \cos \theta d\theta = 9 \int \sin^2 \theta d\theta \\ &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta = \frac{9}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + c \\ &= \frac{9}{2} (\theta - \sin \theta \cdot \cos \theta) + c \\ &= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} \right) + c = \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \cdot \sqrt{9-x^2} + c \end{aligned}$$



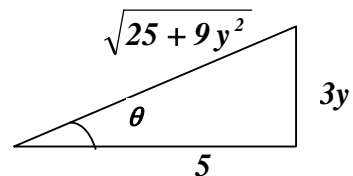
5) let $5t = 3 \sec \theta \Rightarrow 5dt = 3 \sec \theta \cdot \tan \theta d\theta$

$$\begin{aligned} \int \frac{dt}{\sqrt{25t^2-9}} &= \int \frac{\frac{3}{5} \sec \theta \cdot \tan \theta d\theta}{\sqrt{9 \sec^2 \theta - 9}} = \frac{1}{5} \int \sec \theta d\theta \\ &= \frac{1}{5} \ln |\sec \theta + \tan \theta| + c \\ &= \frac{1}{5} \ln \left| \frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3} \right| + c \\ &= \frac{1}{5} \ln |5t + \sqrt{25t^2-9}| + c' \quad \text{where } c' = c - \frac{1}{5} \ln 3 \end{aligned}$$



6) let $3y = 5 \tan \theta \Rightarrow 3dy = 5 \sec^2 \theta d\theta$

$$\begin{aligned} \int \frac{dy}{\sqrt{25+9y^2}} &= \int \frac{\frac{5}{3} \sec^2 \theta d\theta}{\sqrt{25+25 \tan^2 \theta}} = \frac{1}{3} \int \sec \theta d\theta \\ &= \frac{1}{3} \ln |\sec \theta + \tan \theta| + c \\ &= \frac{1}{3} \ln \left| \frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5} \right| + c \\ &= \frac{1}{3} \ln |\sqrt{25+9y^2} + 3y| + c' \quad \text{where } c' = c - \frac{1}{3} \ln 5 \end{aligned}$$



EX - Prove the following formulas:

$$1) \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c \qquad 2) \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$$

Proof.-

$$1) \text{ let } u = a \sin \theta \Rightarrow du = a \cos \theta \cdot d\theta$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a \cos \theta \cdot d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} = \int d\theta = \theta + c = \sin^{-1} \frac{u}{a} + c$$

$$2) \text{ let } u = a \tan \theta \Rightarrow du = a \sec^2 \theta \cdot d\theta$$

$$\int \frac{du}{a^2 + u^2} = \int \frac{a \sec^2 \theta \cdot d\theta}{a^2 + a^2 \tan^2 \theta} = \frac{1}{a} \int d\theta = \frac{1}{a} \theta + c = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$$

Integral involving $ax^2 + bx + c$:

By using the algebraic process called completing the square, we can convert any quadratic: $ax^2 + bx + c$, $a \neq 0$ to the form: $a(u^2 \mp A^2)$ we can then use one of the trigonometric substitutions to write the expression as a times a single square term.

EX – Evaluate:

$$1) \int \frac{dx}{\sqrt{2x - x^2}}$$

$$4) \int \frac{dx}{\sqrt{1 + x - x^2}}$$

$$2) \int \frac{dx}{2x^2 + 2x + 1}$$

$$5) \int \frac{dx}{\sqrt{x^2 - 2x - 8}}$$

$$3) \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

Sol.

$$1) \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 - (x^2 - 2x + 1)}} = \int \frac{dx}{\sqrt{1 - (x - 1)^2}}$$

$$\text{let } x - 1 = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{\cos \theta d\theta}{\sqrt{1 - \sin^2 \theta}} = \int d\theta = \theta + c = \sin^{-1} (x - 1) + c$$

$$2) \int \frac{dx}{2x^2 + 2x + 1} = \frac{1}{2} \int \frac{dx}{x^2 + x + 1/2} = \frac{1}{2} \int \frac{dx}{\left(x + 1/2\right)^2 + 1/4}$$

$$\text{let } x + \frac{1}{2} = \frac{1}{2} \tan \theta \Rightarrow dx = \frac{1}{2} \sec^2 \theta d\theta$$

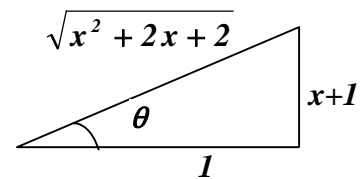
$$\int \frac{dx}{2x^2 + 2x + 1} = \frac{1}{2} \int \frac{\frac{1}{2} \sec^2 \theta d\theta}{\frac{1}{4} \tan^2 \theta + \frac{1}{4}} = \int d\theta = \theta + c = \tan^{-1}(2x + 1) + c$$

$$3) \int \frac{dx}{\sqrt{x^2 + 2x + 2}} = \int \frac{dx}{\sqrt{(x + 1)^2 + 1}}$$

$$\text{let } x + 1 = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\int \frac{dx}{\sqrt{x^2 + 2x + 2}} = \int \frac{\sec^2 \theta d\theta}{\sqrt{\tan^2 \theta + 1}} = \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + c = \ln \left| \sqrt{x^2 + 2x + 2} + x + 1 \right| + c$$



$$4) \int \frac{dx}{\sqrt{1 + x - x^2}} = \int \frac{dx}{\sqrt{5/4 - \left(x - 1/2\right)^2}}$$

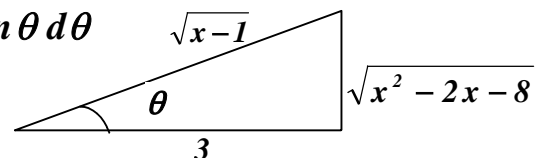
$$\text{let } x - \frac{1}{2} = \frac{\sqrt{5}}{2} \sin \theta \Rightarrow dx = \frac{\sqrt{5}}{2} \cos \theta d\theta$$

$$= \int \frac{\frac{\sqrt{5}}{2} \cos \theta d\theta}{\sqrt{5/4 - 5/4 \sin^2 \theta}} = \int d\theta = \theta + c = \sin^{-1} \left(\frac{2x - 1}{\sqrt{5}} \right) + c$$

$$5) \int \frac{dx}{\sqrt{x^2 - 2x - 8}} = \int \frac{dx}{\sqrt{(x - 1)^2 - 9}}$$

$$\text{let } x - 1 = 3 \sec \theta \Rightarrow dx = 3 \sec \theta \cdot \tan \theta d\theta$$

$$= \int \frac{3 \sec \theta \cdot \tan \theta d\theta}{\sqrt{9 \sec^2 \theta - 9}} = \int \sec \theta d\theta$$



$$= \ln |\sec \theta + \tan \theta| + c = \ln \left| \frac{x - 1}{3} + \frac{\sqrt{x^2 - 2x - 8}}{3} \right| + c$$

$$= \ln \left| x - 1 + \sqrt{x^2 - 2x - 8} \right| + c' \quad \text{where } c' = c - \ln 3$$

Partial fractions:

Success in separating $\frac{f(x)}{g(x)}$ into a sum of partial fractions hinges on two things:-

- 1- The degree of $f(x)$ must be less than the degree of $g(x)$. (If this is not case, we first perform a long division, and then work with the remainder term).
- 2- The factors of $g(x)$ must be known. If these two conditions are met we can carry out the following steps:

Step I - let $x - r$ be a linear factor of $g(x)$. Suppose $(x - r)^m$ is the highest power of $(x - r)$ that divides $g(x)$. Then assign the sum of m partial factors to this factor, as follows:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \dots + \frac{A_m}{(x - r)^m}$$

Do this for each distinct linear factor of $f(x)$.

Step II - let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$. Suppose $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_n x + C_n}{(x^2 + px + q)^n}$$

Do this for each distinct linear factor of $g(x)$.

Step III - set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the sums in decreasing powers of x .

Step IV - equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EX – Evaluate the following integrals:

$$1) \int \frac{2x+5}{x^2-9} dx$$

$$4) \int \frac{\sin x \, dx}{\cos^2 x - 5 \cos x + 4}$$

$$2) \int \frac{x \, dx}{x^2+4x+3}$$

$$5) \int \frac{2x^2-3x+2}{(x-1)^2(x-2)} dx$$

$$3) \int \frac{x^3-x}{(x^2+1) \cdot (x-1)^2} dx$$

$$6) \int \frac{x^3+4x^2}{x^2+4x+3} dx$$

Sol.-

$$1) \int \frac{2x+5}{x^2-9} dx = \int \frac{2x+5}{(x-3) \cdot (x+3)} dx$$

$$\frac{2x+5}{(x-3) \cdot (x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow 2x+5 = A(x+3) + B(x-3)$$

$$\text{at } x=3 \Rightarrow 6A=6+5 \Rightarrow A=\frac{11}{6}$$

$$\text{at } x=-3 \Rightarrow -6B=-6+5 \Rightarrow B=\frac{1}{6}$$

$$\int \frac{2x+5}{x^2-9} dx = \int \left(\frac{11/6}{x-3} + \frac{1/6}{x+3} \right) dx = \frac{11}{6} \ln(x-3) + \frac{1}{6} \ln(x+3) + c$$

$$2) \int \frac{x \, dx}{x^2+4x+3} = \int \frac{x \, dx}{(x+3)(x+1)}$$

$$\frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Rightarrow x = A(x+1) + B(x+3)$$

$$\text{at } x=-3 \Rightarrow A=\frac{3}{2} \text{ and at } x=-1 \Rightarrow B=-\frac{1}{2}$$

$$\int \frac{x \, dx}{x^2+4x+3} = \int \left(\frac{3/2}{x+3} + \frac{-1/2}{x+1} \right) dx = \frac{3}{2} \ln(x+3) - \frac{1}{2} \ln(x+1) + c$$

$$3) \int \frac{x^3 - x}{(x^2 + 1)(x - 1)^2} dx = \int \frac{x(x - 1)(x + 1)}{(x^2 + 1)(x - 1)^2} dx = \int \frac{x^2 + x}{(x^2 + 1)(x - 1)} dx$$

$$\frac{x^2 + x}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} \Rightarrow x^2 + x = (Ax + B)(x - 1) + C(x^2 + 1)$$

$$x^2 + x = (A + C)x^2 + (-A + B)x + (-B + C)$$

$$\left. \begin{array}{l} A + C = 1 \text{ (1)} \\ -A + B = 1 \text{ (2)} \\ -B + C = 0 \text{ (3)} \end{array} \right\} \Rightarrow A = 0, B = 1, C = 1$$

$$\int \frac{x^3 - x}{(x^2 + 1)(x - 1)^2} dx = \int \left(\frac{1}{x^2 + 1} + \frac{1}{x - 1} \right) dx = \tan^{-1} x + \ln(x - 1) + c$$

$$4) \text{ let } y = \cos x \Rightarrow dy = -\sin x dx$$

$$\int \frac{\sin x dx}{\cos^2 x - 5 \cos x + 4} = -\int \frac{dy}{y^2 - 5y + 4} = -\int \frac{dy}{(y - 4)(y - 1)}$$

$$\frac{dy}{(y - 4)(y - 1)} = \frac{A}{y - 4} + \frac{B}{y - 1} \Rightarrow 1 = A(y - 1) + B(y - 4)$$

$$\text{at } y = 4 \Rightarrow A = \frac{1}{3} \text{ and at } y = 1 \Rightarrow B = -\frac{1}{3}$$

$$\int \frac{\sin x dx}{\cos^2 x - 5 \cos x + 4} = -\int \left(\frac{1/3}{y - 4} + \frac{-1/3}{y - 1} \right) dy$$

$$= -\frac{1}{3} \ln(y - 4) + \frac{1}{3} \ln(y - 1) + c = -\frac{1}{3} \ln(\cos x - 4) + \frac{1}{3} \ln(\cos x - 1) + c$$

$$5) \frac{2x^2 - 3x + 2}{(x - 1)^2(x - 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2}$$

$$2x^2 - 3x + 2 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2$$

$$\left. \begin{array}{l} A + C = 2 \text{ (1)} \\ -3A + B - 2C = -3 \text{ (2)} \\ 2A - 2B + C = 2 \text{ (3)} \end{array} \right\} \Rightarrow A = -2, B = -1, C = 4$$

$$\int \frac{2x^2 - 3x + 2}{(x - 1)^2(x - 2)} dx = \int \left(\frac{-2}{x - 1} + \frac{1}{(x - 1)^2} + \frac{4}{x - 2} \right) dx$$

$$= -2 \ln(x - 1) + \frac{1}{x - 1} + 4 \ln(x - 2) + C$$

$$6) \frac{x^3 + 4x^2}{x^2 + 4x + 3} = x - \frac{3x}{(x+3)(x+1)} \quad \begin{array}{r} x^2 + 4x + 3 \overline{) \frac{x^3 + 4x^2}{+ x^3 + 4x^2 + 3x} } \\ \hline -3x \end{array}$$

$$\frac{3x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Rightarrow 3x = A(x+1) + B(x+3)$$

$$\text{at } x = -3 \Rightarrow A = \frac{9}{2} \text{ and at } x = -1 \Rightarrow B = -\frac{3}{2}$$

$$\begin{aligned} \int \frac{x^3 + 4x^2}{x^2 + 4x + 3} dx &= \int \left(x - \frac{9/2}{x+3} + \frac{3/2}{x+1} \right) dx \\ &= \frac{x^2}{2} - \frac{9}{2} \ln(x+3) - \frac{3}{2} \ln(x+1) + c \end{aligned}$$

Rational functions of $\sin x$ and $\cos x$, and other trigonometric integrals:

We assume that $z = \tan \frac{x}{2}$ then $x = 2 \tan^{-1} z$ and $dx = \frac{2}{1+z^2} dz$

Since

$$\begin{aligned} \cos^2 \frac{x}{2} &= \frac{1 + \cos x}{2} \Rightarrow \cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{\sec^2 \frac{x}{2}} - 1 \\ &= \frac{2}{\tan^2 \frac{x}{2} + 1} - 1 = \frac{2}{z^2 + 1} - 1 \Rightarrow \cos x = \frac{1 - z^2}{1 + z^2} \end{aligned}$$

Since

$$\begin{aligned} \sin x &= 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2} = 2 \tan \frac{x}{2} \cdot \frac{1}{\sec^2 \frac{x}{2}} \\ &= 2 \tan \frac{x}{2} \cdot \frac{1}{\tan^2 \frac{x}{2} + 1} \Rightarrow \sin x = \frac{2z}{1 + z^2} \end{aligned}$$

EX – Evaluate:

$$1) \int \frac{dx}{1 + \sin x + \cos x}$$

$$4) \int \frac{3 dx}{2 + 4 \sin x}$$

$$2) \int \frac{dx}{\sin x + \tan x}$$

$$5) \int \sec x dx$$

$$3) \int \frac{dx}{2 + \sin x}$$

$$6) \int \frac{\cos x dx}{1 - \cos x}$$

Sol.-

$$1) \int \frac{dx}{1 + \sin x + \cos x} = \int \frac{\frac{2}{1+z^2} dz}{1 + \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} = \int \frac{dz}{1+z}$$

$$= \ln|1+z| + c = \ln\left|1 + \tan \frac{x}{2}\right| + c$$

$$2) \int \frac{dx}{\sin x + \tan x} = \int \frac{\frac{2}{1+z^2} dz}{\frac{2z}{1+z^2} + \frac{2z}{1-z^2}} = \frac{1}{2} \int \left(\frac{1}{z} - z\right) dz$$

$$= \frac{1}{2} \left[\ln z - \frac{z^2}{2} \right] + c = \frac{1}{2} \left[\ln \tan \frac{x}{2} - \frac{1}{2} \tan^2 \frac{x}{2} \right] + c$$

$$3) \int \frac{dx}{2 + \sin x} = \int \frac{\frac{2}{1+z^2} dz}{2 + \frac{2z}{1+z^2}} = \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

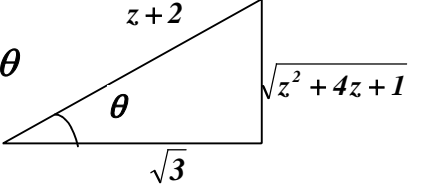
$$\text{let } z + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta \Rightarrow dz = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$$

$$\int \frac{dx}{2 + \sin x} = \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\frac{3}{4} \tan^2 \theta + \frac{3}{4}} = \frac{2}{\sqrt{3}} \int d\theta = \frac{2}{\sqrt{3}} \theta + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2z+1}{\sqrt{3}} \right) + c = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \right) + c$$

$$\begin{aligned}
 4) \int \frac{3 \, dx}{2+4 \sin x} &= \frac{3}{2} \int \frac{dx}{1+2 \sin x} = \frac{3}{2} \int \frac{\frac{2}{z^2+1} dz}{1+2 \frac{2z}{z^2+1}} = 3 \int \frac{dz}{z^2+4z+1} \\
 &= 3 \int \frac{dz}{(z+2)^2-3} = \int \frac{dz}{(\frac{z+2}{\sqrt{3}})^2-1}
 \end{aligned}$$

$$\text{let } \frac{z+2}{\sqrt{3}} = \sec \theta \Rightarrow dz = \sqrt{3} \sec \theta \cdot \tan \theta \, d\theta$$

$$\int \frac{3 \, dx}{2+4 \sin x} = \int \frac{\sqrt{3} \sec \theta \cdot \tan \theta \, d\theta}{\sec^2 \theta - 1} = \sqrt{3} \int \frac{\sec \theta}{\tan \theta} d\theta$$


$$= \sqrt{3} \int \csc \theta \, d\theta = -\sqrt{3} \ln |\csc \theta + \cot \theta| + c$$

$$= -\sqrt{3} \ln \left| \frac{z+2}{\sqrt{z^2+4z+1}} + \frac{\sqrt{3}}{\sqrt{z^2+4z+1}} \right| + c = -\sqrt{3} \ln \left| \frac{\tan \frac{x}{2} + 2 + \sqrt{3}}{\sqrt{\tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1}} \right| + c$$

$$5) \int \sec x \, dx = \int \frac{1+z^2}{1-z^2} \cdot \frac{2}{1+z^2} dz = 2 \int \frac{1}{(1-z)(1+z)} dz$$

$$\frac{1}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z} \Rightarrow A(1+z) + B(1-z) = 1$$

$$\text{at } z=1 \Rightarrow A = \frac{1}{2} \quad \text{and} \quad \text{at } z=-1 \Rightarrow B = \frac{1}{2}$$

$$\int \sec x \, dx = 2 \int \left(\frac{1/2}{1-z} + \frac{1/2}{1+z} \right) dz = -\ln(1-z) + \ln(1+z) + c$$

$$= \ln \left(1 + \tan \frac{x}{2} \right) - \ln \left(1 - \tan \frac{x}{2} \right) + c = \ln \left| \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right| + c$$

$$\text{By substituting } \tan \frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \quad \text{implies}$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + c$$

$$6) \int \frac{\cos x \, dx}{1 - \cos x} = \int \frac{\frac{1 - z^2}{1 + z^2}}{1 - \frac{1 - z^2}{1 + z^2}} \cdot \frac{2}{1 + z^2} dz = \int \frac{1 - z^2}{(1 + z^2) z^2} dz$$

$$\frac{1 - z^2}{(1 + z^2) z^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz + D}{1 + z^2} \Rightarrow$$

$$Az + Az^3 + B + Bz^2 + Cz^3 + Dz^2 = 1 - z^2$$

$$\left. \begin{array}{l} A + C = 0 \quad \dots\dots(1) \\ B + D = -1 \quad \dots(2) \\ A = 0 \quad \dots\dots\dots(3) \\ B = 1 \quad \dots\dots\dots(4) \end{array} \right\} \Rightarrow A = 0, B = 1, C = 0, D = -2$$

$$\int \frac{\cos x \, dx}{1 - \cos x} = \int \left(\frac{1}{z^2} - \frac{2}{z^2 + 1} \right) dz = -\frac{1}{z} - 2 \tan^{-1} z + c$$

$$= -\frac{1}{\tan \frac{x}{2}} - 2 \cdot \frac{x}{2} + c = -\cot \frac{x}{2} - x + c$$