Infinite Series

An infinite series is given by the terms of an infinite sequence, added together. For example, we could take the infinite sequence

$$\left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right\}$$

Then the corresponding example of an infinite series would be given by all of these terms added together

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

So we would have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

If $\sum a_n$ is a positive series, then either $\sum a_n$ converges to a positive number, or $\sum a_n$ diverges to infinity.

Divergence Test:

$$\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ then } \sum a_n \text{ diverges.}$$

Do not misuse this test. This test only says that a series is guaranteed to diverge if the series terms don't go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge!

Again, recall the following two series,

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges}$$

One of the more common mistakes that students make when they first get into series is to assume that if $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ will converges. There is just no way to guarantee this so be careful.

Let's take a quick look at an example of how this test can be used.

Example 1: Determine if the following series is convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{n+2}{n}$ 2. $\sum_{n=0}^{\infty} \frac{4n^2-n^3}{5+2n^3}$

Solution

1. $\lim_{n\to\infty} \frac{n+2}{n} = 1 \neq 0$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

2. $\lim_{n\to\infty} \frac{4n^2-n^3}{5+2n^3} = \lim_{n\to\infty} \frac{8n-3n^2}{6n^2} = \lim_{n\to\infty} \frac{8-6n}{12n} = \lim_{n\to\infty} \frac{-6}{12} = -\frac{1}{2} \neq 0$

By the Divergence Test the series diverges.

Special Series

Geometric Series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

1. If $|r| < 1$ then the geometric series converges to the sum $s_n = \frac{a}{1-r}$
2. If $|r| \geq 1$ then the geometric series diverges to ∞ .

Example 2: Determine if the following series converge or diverge. If they converge give the value of the series.

1. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ 2. $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ 3. $\sum_{n=1}^{\infty} 9^{1-n}\, 2^{n+1}$

Solution

1. The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because $r = \frac{1}{2} < 1$

$$a = \frac{1}{2}, \qquad \text{the sum is } s_n = \frac{1/2}{1-(1/2)} = 1.$$

2. The series $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ diverges to ∞ because $r = \frac{3}{2} > 1$

3. The series $\sum_{n=1}^{\infty} 9^{1-n}\, 2^{n+1} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{9^{n-1}} = 4 + \frac{2^3}{9} + \frac{2^4}{9^2} + \frac{2^5}{9^3} + \cdots$

converges because $r = \frac{2}{9} < 1$

$$a = 4, \qquad \text{the sum is } s_n = \frac{4}{1-(2/9)} = 4 \times \frac{9}{7} = \frac{36}{7}$$

The p – series

If p is a real constant, the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

1. converges if $p > 1$.   2. diverges if $p \leq 1$.

Example 3: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $p = 2 > 1$

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges to ∞ because $p = \frac{1}{2} < 1$

Tests for converges of series

1. Integral Test

Let the function $f(x) = a_n(x)$ be continuous, positive and decreasing then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Example 4: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ 2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ 3. $\sum_{n=1}^{\infty} \frac{1}{n^2-n-2}$

1. $\int_1^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ converge

2. $\int_1^{\infty} \frac{xdx}{x^2+1} = \frac{1}{2}\ln|x^2+1|\Big|_1^{\infty} = \infty$ diverge

3. $\frac{1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$: $x = 2 \Rightarrow A = \frac{1}{3}$ and $x = -1 \Rightarrow B = -\frac{1}{3}$

$$\int_1^{\infty} \frac{dx}{x^2-x-2} = \frac{1}{3}\int_1^{\infty}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\Bigg|_1^{\infty}$$

$$= \frac{1}{3}\ln\left|\frac{1-\frac{2}{x}}{1+\frac{1}{x}}\right|\Bigg|_1^{\infty} = \frac{1}{3}\ln 2 \quad \text{converge}$$

2. Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non – negative terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

If 1. $L < 1$ then the series converges

2. $L > 1$ then the series diverges

3. $L = 1$ then this test is inconclusive

Example 5: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ 2. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ 3. $\sum_{n=1}^{\infty} \frac{3}{2n+5}$ 4. $\sum_{n=1}^{\infty} \frac{5^n}{n\,3^n}$

1. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3} = \frac{1}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1 \quad \text{converges}$$

2. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1) \times n!}{2^n \times 2} \times \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1 \quad \text{diverges}$$

3. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{2(n+1)+5} \times \frac{2n+5}{3}$$

$$= \lim_{n\to\infty} \frac{2n+5}{2n+7} = 1. \quad \text{The test is inconclusive .}$$

So we try another test $\int_1^{\infty} \frac{3dx}{2x+5} = \frac{3}{2} \ln|2x+5| \Big|_1^{\infty} = \infty$ diverges

4. $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)\,3^{n+1}} \times \frac{n\,3^n}{5^n}$$

$$= \lim_{n\to\infty} \frac{5n}{3(n+1)} = \frac{5}{3} > 1 \quad \text{diverges}$$

3. Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non – negative terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

If 1. $L < 1$ then the series converges

2. $L > 1$ then the series diverges

3. $L = 1$ then this test is inconclusive

Example 6: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+1}\right)^n$ 2. $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ 3. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ 4. $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$

1. $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3n+2}{4n+1} = \frac{3}{4} < 1$ converges

2. $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$ converges

3. $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^2}} = \frac{2}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{2}{1} = 2 > 1$ diverges

4. $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n-1}{n} = 1.$ The test is inconclusive .

So we try another test $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} \neq 0$ diverges

Exercises

Find the sum of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$ 2. $\sum_{n=1}^{\infty} \frac{5^n}{3^{2n}}$ 3. $\sum_{n=1}^{\infty} \frac{4}{3^{n+1}}$□

□

Determine whether the series converges or diverges. Give reasons for your answers.

4. $\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$

5. $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$

6. $\sum_{n=1}^{\infty} \frac{(n-1)!}{n\,!}$

7. $\sum_{n=1}^{\infty} \frac{11^n}{3^{2n}}$

8. $\sum_{n=1}^{\infty} ne^{-n^2}$

9. $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

10. $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^{-n}$

11. $\sum_{n=1}^{\infty} \frac{1}{(3n+1)^3}$

12. $\sum_{n=1}^{\infty} \frac{(n+3)!}{2^n n!}$