D'Alembert's Solution for Wave Equation

In this lecture we discuss the one dimensional wave equation. We review some of the physical situations in which the wave equations describe the dynamics of the physical system.

Suppose we have the wave equation $u_{tt} - c^2 u_{xx} = 0$ and we wish to solve it given the conditions $u(x,0) = F(x)$ and $u_t(x,0) = G(x)$.

We change variables to $r = x + ct$ and $s = x - ct$. The general solution of this PDE is $u(x,t) = F(x + ct\,) + G(x - ct)$

Example 1: Solve the initial value problem $u_{tt} - 4u_{xx} = 0$

with $u(x,0) = 0$ and $u_t(x,0) = \tan x$

Solution :

$$u(x,t) = F(x + 2t\,) + G(x - 2t)$$

$$u(x,0) = 0 \quad \Rightarrow \quad 0 = F(x + 0\,) + G(x - 0)$$

$$F(x\,) + G(x) = 0 \qquad \cdots (1)$$

$$u_t(x,t) = 2\,F'(x + 2t\,) - 2G'(x - 2t)$$

$$u_t(x,0) = \tan x \;\Rightarrow\; 2\,F'(x) - 2G'(x) = \tan x$$

$$2\,F(x) - 2G(x) = -\ln\cos(x) \qquad \cdots\ (2)$$

$$2 \times equ(1) + equ(2) \;\Rightarrow\; 4\,F(x) = -\ln\cos(x)$$

$$F(x) = -\frac{1}{4}\ln\cos(x) \qquad \text{and} \;\; G(x) = \frac{1}{4}\ln\cos(x)$$

So $F(x + 2t\,) = -\frac{1}{4}\ln\cos(x + 2t\,)$ and $G(x - 2t\,) = \frac{1}{4}\ln\cos(x - 2t\,)$

Then $u(x,t) = \frac{1}{4}\ln\cos(x - 2t\,) - \frac{1}{4}\ln\cos(x + 2t\,)$

Or $u(x,t) = \frac{1}{4}\ln\frac{\cos(x - 2t\,)}{\cos(x + 2t\,)}$

Example 2: Solve the initial value problem $u_{tt} - a^2 u_{xx} = 0$

with $u(x,0) = 0$ and $u_t(x,0) = \dfrac{1}{1+x^2}$

Solution :

$$u(x,t) = F(x+at) + G(x-at)$$

$$u(x,0) = 0 \quad \Rightarrow \quad 0 = F(x+0) + G(x-0)$$

$$F(x) + G(x) = 0 \qquad \cdots (1)$$

$$u_t(x,t) = a\,F'(x+at) - aG'(x-at)$$

$$u_t(x,0) = \frac{1}{1+x^2} \quad \Rightarrow \quad aF'(x) - aG'(x) = \frac{1}{1+x^2}$$

$$aF(x) - aG(x) = \tan^{-1} x \qquad \cdots (2)$$

$$a \times equ(1) + equ(2) \quad \Rightarrow \quad 2a\,F(x) = \tan^{-1} x$$

$$F(x) = \frac{1}{2a}\tan^{-1} x \qquad \text{and} \quad G(x) = -\frac{1}{2a}\tan^{-1} x$$

So $F(x+at) = \dfrac{1}{2a}\tan^{-1}(x+at)$ and $G(x-at) = \dfrac{1}{2a}\tan^{-1}(x-at)$

Then $u(x,t) = \dfrac{1}{2a}\tan^{-1}(x+at) - \dfrac{1}{2a}\tan^{-1}(x-at)$

H.W: Solve the initial value problems

1. $u_{tt} - a^2 u_{xx} = 0$ with $u(x,0) = 0$ and $u_t(x,0) = \sin x$
2. $u_{tt} - 9u_{xx} = 0$ with $u(x,0) = 0$ and $u_t(x,0) = e^{2x}$