Differentiation

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in chapter two. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

Tangents and the Derivative at a Point

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_o, f(x_o))$, we calculate the slope of the secant through P and a nearby point $Q(x_o + h, f(x_o + h))$.We then investigate the limit of the slope as $h \to 0$ (Figure 1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

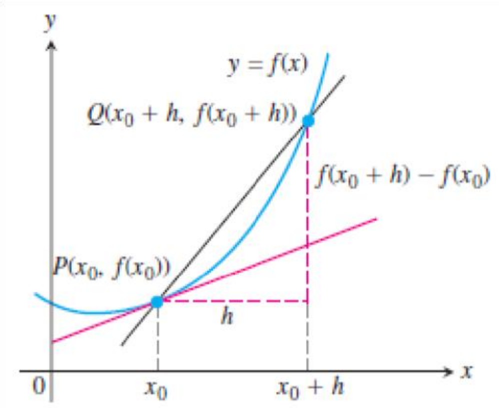

Figure 1

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal – ¼?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes? **Solution**:

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{a + h} - \dfrac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a + h)}{a(a + h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a + h)} = \lim_{h \to 0} \frac{-1}{a(a + h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h \to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 2).

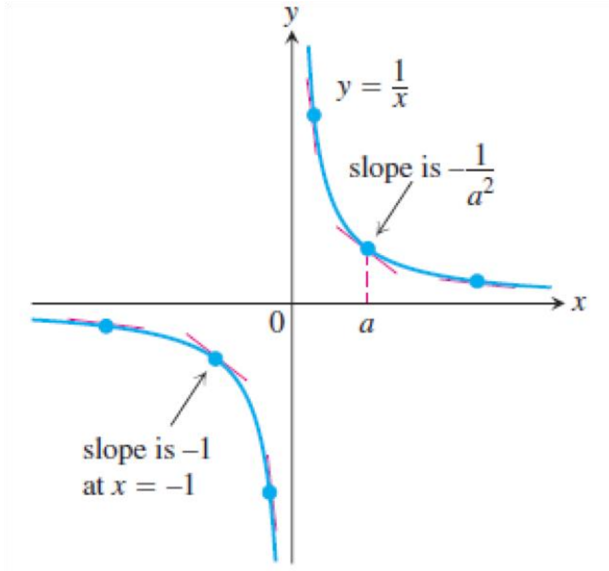

Figure 2

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/ a^2$. It will be provided that

$$- 1/a^2 = - 1/4$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope -1/4 at the two points (2, 1/2) and (-2, -1/2) (Figure 3).

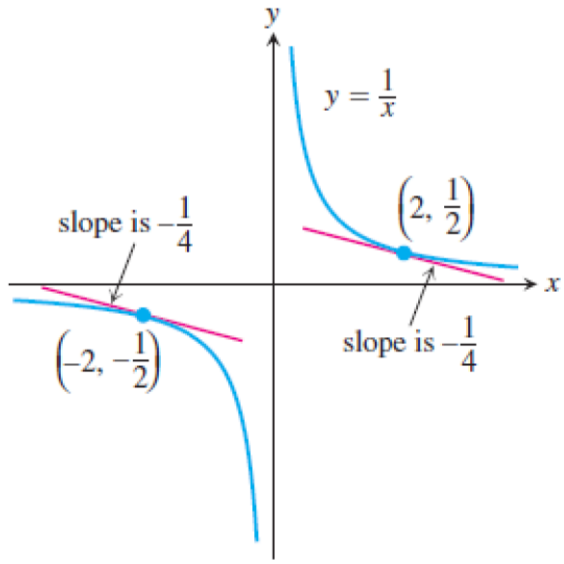

Figure 3

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal.

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \qquad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Summary

All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx}f(x)$$

There are many ways to denote the derivative of a function *y = f(x),* where the independent variable is *x* and the dependent variable is y. Some common alternative notations for the derivative are:

$$f(x) = y = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

Example 2: by using the definition of the derivative, find *dy/dx* of the function $y = 5x^3 + 8x^2 - 3x + 4$ **Solution:**

$f(x) = 5x^3 + 8x^2 - 3x + 4$ $f(x + \Delta x) = 5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5(x^3 + 3x^2\,\Delta x + 3x\Delta x^2 + \Delta x^3) + 8(x^2 + 2x\Delta x + \Delta x^2) - 3x - 3\Delta x + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5x^3 + 15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 8x^2 + 16x\Delta x + 8\Delta x^2 - 3x - 3\Delta x - 5x^3 - 8x^2 + 3x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 16x\Delta x + 8\Delta x^2 - 3\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} 15x^2 + 15x\Delta x + 5\Delta x^2 + 16x + 8\Delta x - 3$$

$= 15\,x^2 + 16\,x - 3$

Differentiation Rules

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 3: $\frac{d}{dx} 5 = 0$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx} x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Example 4: Differentiate the following equations:

(a) x^3, (b) $x^{2/3}$, (c) $\sqrt{}$, (d) $\frac{1}{x^4}$, (e) $x^{-4/3}$, (f) $\sqrt{x^{2+\pi}}$

Solution:

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example 5: Differentiate the equation $y = 3x^2$

Solution: $\frac{dy}{dx}(3x^2) = 3 * 2x = 6x$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 6: Find the derivative of the polynomial $y = x^3 + (4/3)x^2 - 5x + 1$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$

$= 3x^2 + (4/3)*2x - 5 + 0$

$= 3x^2 + (8/3)*2x - 5$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7: find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution:

(a) From the Product Rule we find:

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 8: find the derivative of $y = \frac{t^2-1}{t^{3+1}}$ **Solution**: apply the Quotient Rule:

$$\frac{dy}{dt} = \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

Example 9: Find an equation for the tangent to the curve $y = x + 1/x$ at x = 2. **Solution**:

At $x = 2$:

$y = 2 + ½ = 5/2$ point

(2, 5/2)

$y = x + \frac{1}{x}$

$\frac{dy}{dx} = 1 + \frac{(x)(0)-(1)(1)}{x^2}$

$= 1 + \frac{-1}{x^2}$

At $x = 2$:

$\frac{dy}{dx} = m = 1 - \frac{1}{4} = ¾$

$(y - y_1) = m\,(x - x_1)$

$(y - 5/2) = 3/4\,(x - 2)$

$(2y - 5)/2 = 3(x - 2)/4$

$8y - 20 = 6(x - 2)$

$8y - 20 = 6x - 12$

$8y - 6x - 8 = 0$ y

$= 6/8\,x + 1$

Example 10: Find the point on the curve $y = x^3 + x^2 - 1$ where the tangent is parallel to the x- axis.

Solution:

Slope = $dy/dx = 3x^2 + 2x$

When the tangent is parallel to the x-axis, $m = 0$.

$3x^2 + 2x = 0$ x

$(3x + 2) = 0$

$x = 0$ or $3x + 2 = 0$ $x = -2/3$ at

$x = 0$ $y = -1$

$P_1\,(0, -1)$

At x = -2/3 y = -23/27
P_2 = (-2/3, -23/27)

Example 11: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?
Solution: The horizontal tangents, if any, occur where the slope dy/dx is zero. We have: $dy/dx = d/dx\,(x^4 - 2x^2 + 2)$

$$= 4x^3 - 4x$$

Now solve the equation $dy/dx = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1$$

The curve $y = x4 - 2x2 + 2$ has horizontal tangents at x = 0, 1 and -1. The corresponding points on the curve are (0, 2), (1, 1) and (-1, 1)

Derivatives of Trigonometric Functions

Derivative of the Sine Function

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

> **The derivative of the cosine function is the negative of the sine function:**
>
> $$\frac{d}{dx}(\cos x) = -\sin x.$$

Example 12: find the derivatives of the following functions:

(a) $y = \sin x \cos x$

(b) $y = \frac{cosx}{1-sinx}$

Solution:

(a)

$y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

(b)

$y = \frac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$
$$= \frac{1}{1 - \sin x}$$

3.6.2 Derivatives of the Other Basic Trigonometric Functions

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 13: derive the following equations:

$y = \cos x \tan 3x$

$y = \tan\sqrt{3x}$ y

$= sin^2(\frac{1}{x})$

Solution:

$dy/dx = \cos x\ (\sec^2 3x * 3) + \tan 3x\ (-\sin x)$

$= 3 \cos x \sec^2 3x - \sin x \tan 3x$

$dy/dx = \sec^2 (3x)^{1/2} * ½ (3x)^{-1/2} * 3$

$= \frac{3}{2\sqrt{3x}}\ sec^2\sqrt{3x}$

$y = (\sin\left(\frac{1}{x}\right))^2$

$dy/dx = 2\ (\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (-1 \times x^{-2}))\ dy/dx$

$= 2(\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (\frac{-1}{x^2})$

$= \frac{-2}{x^2} \sin\frac{1}{x} \cos\frac{1}{x}$

The Chain Rule

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 4).

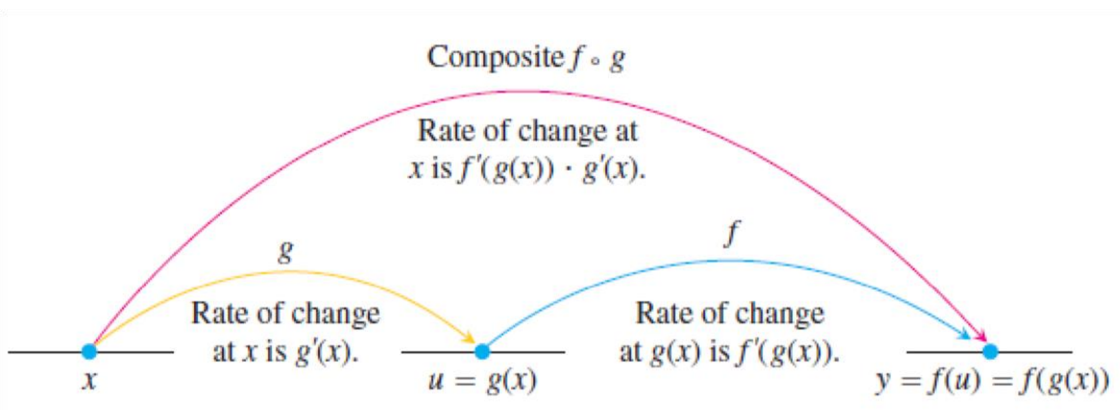

Figure 4

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example 15: If $y = x^2 + 2x + 1$, $x = 3u^2 + 1$, find dy/du.

Solution:
Method 1: substitute *x* function in *y* function:
$y = (3u^2 + 1)^2 + 2(3u^2 + 1) + 1 \quad = 9u^4 + 6u^2 +$
$1 + 6u^2 + 2 + 1$
$= 9u^4 + 12u^2 + 4$
$dy/dx = 36u^3 + 24u$

Method 2: Chain Rule
$dy/du = dy/dx * dx/du$ dy/dx
$= 2x + 2, \quad dx/du = 6u$
$dy/du = (2x + 2)(6u) = [\,2(3u^2 + 1) + 2]\,(6u)$
$= (6u^2 + 4)(6u)$
$= (36u^3 – 24u)$

Example 16: Find *dy/dt* for $y = \sin(t^2 + 6)$ by using Chain Rule **Solution**:
Let $y = \sin u$ and $u = t^2 + 6$
$dy/dt = dy/du * du/dt$ dy/du
$= \cos u, \quad du/dt = 2t$ dy/dx
$= \cos u * 2t \qquad = \cos(t^2$
$+ 6) * 2t$
$= 2t \cos(t^2 + 6)$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example 17: Find the derivative of function $g(t) = \tan(5 - \sin 2t)$.

Solution:

Notice here that the tangent is a function of $5 - \sin 2t$ whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule:

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

Example 18: Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution We find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\ &= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \\ &= -3(1 - 2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1 - 2x)^4}. \end{aligned}$$

At any point (x, y) on the curve, $x \neq ½$ and the slope of the tangent line is :

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4}$$

the quotient of two positive numbers.

Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0,\ y^2 - x = 0 \text{ or } x^2 + y^2 - 25 = 0.$$

These equations define an **implicit** relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by **implicit differentiation**. This section describes the technique.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example 19:

(a) If $x^2 + y^2 = 25$, find dy/dx.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4). **Solution**:

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that *y* is a function of *x* and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus $$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for *dy/dx:*

$$dy/dx = -\,x/y$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$dy/dx = -3/4$$

Example 20: Find *dy/dx* if $y^2 = x^2 + \sin xy$.

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$ Differentiate both sides with respect to x ...

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}\left(xy\right)$$ ... treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx.

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx.

Derivatives of Higher Order

Example21: Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution:
To start, we differentiate both sides of the equation with respect to x in order to find $y = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$ Treat y as a function of x.

$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0$$ Solve for y'.

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$ ■

Example 22: Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 5).

Solution: The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve:

$$2^3 + 4^3 - 9\,(2)\,(4) = 8 + 64 - 72 = 0$$

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for *dy/dx*:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0 \qquad \text{Differentiate both sides with respect to } x.$$

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0 \qquad \text{Treat } xy \text{ as a product and } y \text{ as a function of } x.$$

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}. \qquad \text{Solve for } dy/dx.$$

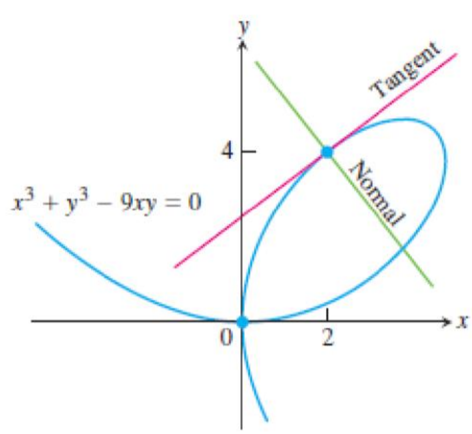

Figure 5

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

Parametric Equations

If $x = f(t)$ and $y = g(t)$, then these equations are called parametric equations and the variable t is called parameter.

From Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\quad \frac{dy}{dx} = \frac{dy}{\frac{\frac{du}{dx}}{du}}$

$x = f(t), y = g(t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ the 1 st derivative for parametric equation

For second derivative:

$\frac{d^2y}{dx^2} = \frac{dy/dt}{dx/dt}, \; y = \frac{dy}{dx}$

Or $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$ the 2nd derivative for parametric equation

Example 23: if $y = 2t^3 + 3$, $x = t/(t-1)$, find dy/dx

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$dy/dt = 6t^2$

dx/dt = $\frac{(t-1)(1)-t(1)}{(t-1)^2} = \frac{-1}{(t-1)^2}$

dy/dx = $\frac{6t^2}{\frac{-1}{(t-1)^2}}$

dy/dx = $-6t^2 (t-1)^2$

Example 24: If a point traces the circle $x^2 + y^2 = 25$ and if *dx/dt* = 4 when the point reaches (3, 4). Find *dy/dt*

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx\ /\ dt}$$

$x^2 + y^2 = 25$ *2x + 2y (dy/dx)* = 0   *dy/dx* = - *x/y* At point (3, 4) *dy/dx* = - 3/4

-3/4 = $\frac{dy/dt}{4}$ *dy/dt*

= - 3

Example 25: If *x = cos 3t* , *y = sin*^{2} *3t*, find *dy/dx*, d^2y/dx^2 **Solution**:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

dy/dt = 2 sin 3t (cos 3t).3 = 6 sin 3t cos 3t dx/dt

= - *sin 3t .3 = -3 sin 3t*

$\frac{dy}{dx} = \frac{6\ sin3t\ cos3t}{-3\ sin3t}$ = -*2 cos 3t* = - *2x*

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-2\cos 3t)$ = *-2 (- sin 3t).3*

$\frac{d^2y}{dx^2} = \frac{-2(-\sin 3t).3}{-3\sin 3t}$ = -2

Or *dy/dx* = - *2x*

d^2y/dx^2 = *-2* when *dy/dx* with respect to *x.*