CHAPTER 1

PRELIMINARIES

1.1 Real Numbers and the Real Line

Calculus is based on the real number system. Real numbers are numbers that can be expressed as decimals.
We distinguish three special subsets of real numbers:

1. The **natural numbers**, namely 1, 2, 3, 4,…
2. The **integers**, namely 0, ±1, ±2, ±3,…
3. The **rational numbers**, which are ratios of integers. These numbers can be expressed in the form of a function *m*/*n,* where *m* and *n* are integers and $n \neq 0$. Examples are:

$$\frac{1}{2}, \; -\frac{5}{3} = \frac{-5}{3} = \frac{5}{-3}, \; \frac{200}{13}, \; 67 = \frac{67}{1}$$

(Recall that division by is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.)

The real numbers can be represented geometrically as points on a number line called the **real line,** as in Figure 1.1.

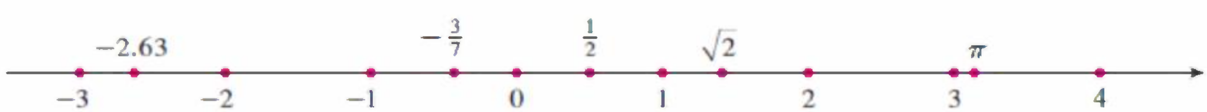

Figure 1.1

1.1.1 Intervals

Certain sets (or a subset) of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from to consists of all numbers between *a* and *b* is denoted by the symbol (*a, b*). Using set-builder notation, we can write:

$$(a, b) = \{x \mid a < x < b\}$$

(which is read "(a, b) is the set of x such that x is an integer and $a < x < b$.)

Notice that the endpoints of the interval -namely, a and b- are excluded. This is indicated by the round brackets and by the open dots in Table 1.1. The **closed interval** from a to b is the set

$$[a, b] = \{x|\ a \leq x \leq b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in table 1.1. It is also possible to include only one endpoint in an interval, as shown in Table 1.1.

Table 1.1

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

1.1.2 Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

The following useful rules can be derived from them, where the symbol => means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Example 1: Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \geq 5$

Solution:

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \quad \text{Add 1 to both sides.}$$

$$x < 4 \quad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval (-∞, 4) (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \quad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \quad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \quad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \quad \text{Divide by 7.}$$

The solution set is the open interval (-3/7, ∞) (Figure 1.1b).

The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$ because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$ and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval (1, 11/5] (Figure 1.1c)

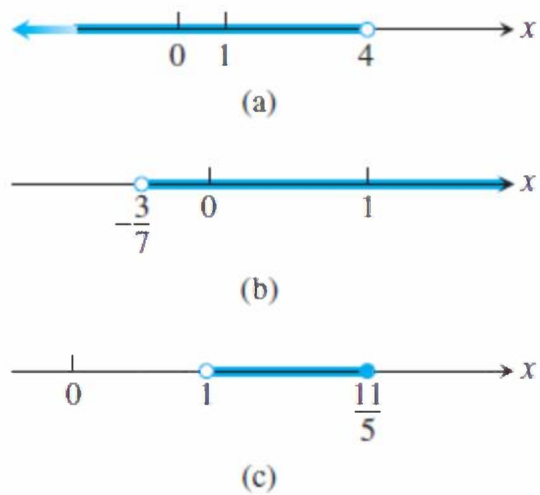

Figure 1.2

1.1.3 Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is the distance from x to 0 on the real number line. Distances are always positive or 0, so we have

$$|x| \geq 0 \quad \text{for every number } x$$

Or it can be defined by the formula:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 2:

$|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $|-|a|| = |a|$

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure 1.2).

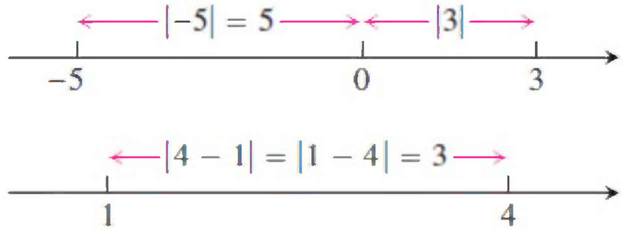

Figure 1.3

The absolute value has the following properties:

Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Example 3:

$$|-3+5| = |2| = 2 < |-3| + |5| = 8$$
$$|3+5| = |8| = |3| + |5|$$
$$|-3-5| = |-8| = 8 = |-3| + |-5|$$

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values:

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

The inequality $|x| < a$ *says* that the distance from x to 0 is less than the positive number a. This means that x must lie between $-a$ and a, as we can see from Figure 1.4.

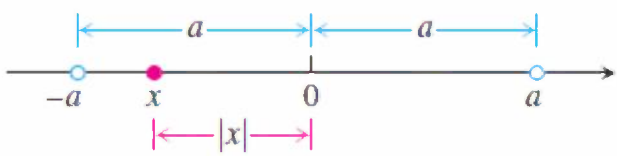

Figure 1.4

Example 4: Solve the equation $|2x - 3| = 7$ **Solution**:

By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$

Example 5: Solve the inequality | −|

Solution We have

$$\left|5-\frac{2}{x}\right|<1 \Leftrightarrow -1<5-\frac{2}{x}<1 \qquad \text{Property 6}$$

$$\Leftrightarrow -6<-\frac{2}{x}<-4 \qquad \text{Subtract 5.}$$

$$\Leftrightarrow 3>\frac{1}{x}>2 \qquad \text{Multiply by } -\frac{1}{2}.$$

$$\Leftrightarrow \frac{1}{3}<x<\frac{1}{2}. \qquad \text{Take reciprocals.}$$

(The symbol <=> is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by.")
The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval (1/3, 1/2).

1.2 Lines, Circles, and Parabolas

1.2.1 Coordinate Geometry and Lines

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin *O* on each line. Usually one line is horizontal with positive direction to the right and is called the ***x*axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. Any point ***P*** in the plane can be located by a unique ordered pair of numbers as follows:
Draw lines through ***P*** perpendicular to the *x*- and *y*-axes. These lines intersect the axes in points with coordinates and as shown in Figure 1.5. Then the point *P* is assigned the ordered pair (*a, b*). The first number *a* is called the ***x*-coordinate** (or **abscissa**) of *P*; the second number *b* is called the ***y*-coordinate** (or **ordinate**) of *P*. We say that *P* is the point with coordinates (*a, b*), and we denote the point by the symbol *P* (*a, b*). Several points are labeled with their coordinates in Figure 1.6.