Solving Systems of Nonlinear Equations

Newton's Method for Non-linear system

The Determinant and Inverse of Matrix

Definition 1: Suppose that A is a square matrix.

1. If $A = [a]$ is a 1×1 matrix then $|A| = a$.
2. If A is an $n \times n$ matrix, with $n > 1$ the **minor** M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.
3. The **cofactor** C_{ij} associated with M_{ij} is defined by $C_{ij} = (-1)^{i+j}M_{ij}$.
4. The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} (-1)^{i+j}M_{ij} \qquad for\ i = 1,2,\dots,n$$

Or

$$|A| = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} (-1)^{i+j}M_{ij} \qquad for\ j = 1,2,\dots,n \qquad \blacksquare$$

Example 1: Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 3 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix}$$

using the row or column with the most zero entries.

Solution: To compute determinant of A, it is easiest to use the fourth column

$$|A| = a_{14}A_{14} + a_{24}A_{24} + a_{34}A_{34} + a_{44}A_{44} = 5A_{34} = -5M_{34}$$

$$|A| = -5\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{vmatrix}$$

$$= -5\left\{2\begin{vmatrix} -2 & 7 \\ -6 & 8 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 7 \\ 6 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & -2 \\ 6 & -6 \end{vmatrix}\right\} = -30$$

Definition 2: The inverse of the square matrix can be defined by:

$$A^{-1} = \frac{1}{|A|}\left(C_{ij}\right)^T = \frac{1}{|A|}\left(C_{ji}\right) = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} \qquad \blacksquare$$

C_{ij} is the cofactor matrix defined above.

Example 2: Find the inverse of the following matrices:

1. $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

2. $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution:

1.

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{3 \times 2 - 1 \times 2}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$$

2.

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 1\big(2 \times 2 - (-1 \times -1)\big) - (-1)\big((-1 \times 2) - (-1 \times 0)\big) + 0 = 3 - 2 = 1$$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 1\big(2 \times 2 - (-1 \times -1)\big) = 3$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -1\big(-1 \times 2 - (-1 \times 0)\big) = 2$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1\big((-1 \times -1) - (2 \times 0)\big) = 1$$

$c_{21} = 2, \quad c_{22} = 2, \quad c_{23} = 1, \quad c_{31} = 1, \quad c_{32} = 1 \quad c_{33} = 1$

$$B^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

System of Nonlinear Equations

A system of nonlinear equations has the form:

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ \vdots \qquad & \quad \vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0, \end{aligned}$$

where each function f_i can be thought of as mapping a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^t$ of the n-dimensional space $\mathbb{R}^n$ into the real line $\mathbb{R}$.
This system of n nonlinear equations in n unknowns can also be represented by defining a function **F** mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ as

$$\mathbf{F}(x_1, x_2, \dots, x_n) = \left(f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n)\right)^t.$$

If vector notation is used to represent the variables $x_1, x_2, \dots, x_n$ then system of equations above assumes the form

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

The functions $f_1, f_2, \dots, f_n$ are called the **coordinate functions** of **F**.

Example 3: Place the following 3 x 3 nonlinear system in the form **F**(**x**) = **0**.

$$3x_1 - cos(x_2x_3) - \frac{1}{2} = 0$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0$$

$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

Solution: Define three coordinate functions $f_1, f_2\ and\ f_3$ from $\mathbb{R}^3$ to $\mathbb{R}$ as

$$f_1(x_1, x_2, x_3) = 3x_1 - cos(x_2x_3) - \frac{1}{2}$$

$$f_2(x_1,x_2,x_3) = x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06$$

$$f_3(x_1,x_2,x_3) = e^{-x_1x_2} + 20x_3 + \frac{10\pi-3}{3}$$

We define $\mathbf{F}$ from $\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x_1,x_2,x_3)\\ &= \big(f_1(x_1,x_2,x_3),\ f_2(x_1,x_2,x_3), f_3(x_1,x_2,x_3)\big)^t\\ &= \Big(3x_1 - cos(x_2x_3) - \tfrac{1}{2},\ x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06,\\ &\quad e^{-x_1x_2} + 20x_3 + \tfrac{10\pi-3}{3}\Big)^t \qquad \blacksquare\end{aligned}$$

Jacobian Matrix

Define the Jacobian matrix $J(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}$$

The function $\boldsymbol{G}$ is defined by:

$$\boldsymbol{G}(\mathbf{x}) = \mathbf{x} - J(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$$

$$\mathbf{x}^{(k)} = \boldsymbol{G}\big(\mathbf{x}^{(k-1)}\big) - J\big(\mathbf{x}^{(k-1)}\big)^{-1}\,\mathbf{F}\big(\mathbf{x}^{(k-1)}\big)$$

This is called **Newton's method for nonlinear systems.** A weakness in Newton's method arises from the need to compute and inverse of the matrix $J(\mathbf{x})$ at each step.

In this method we firstly start with an initial approximation vector $\mathbf{x}^{(0)}$, then we find the Jacobian matrix $J\big(\mathbf{x}^{(0)}\big)$ and $\mathbf{F}\big(\mathbf{x}^{(0)}\big)$ and solve the system

$$J\big(\mathbf{x}^{(0)}\big)\boldsymbol{y}^{(0)} = -F\big(\mathbf{x}^{(0)}\big)$$

$$\boldsymbol{y}^{(0)} = -J\left(\mathbf{x}^{(0)}\right)^{-1} F\left(\mathbf{x}^{(0)}\right)$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \boldsymbol{y}^{(0)}$$

To generalize the situation for every $k \geq 1$

$$\boldsymbol{y}^{(k-1)} = -J\left(\mathbf{x}^{(k-1)}\right)^{-1} F\left(\mathbf{x}^{(k-1)}\right)$$

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + \boldsymbol{y}^{(k-1)}$$

$$J\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\mathbf{x}^{(k-1)} & \frac{\partial f_1}{\partial x_2}\left(\mathbf{x}^{(k-1)}\right) & \cdots & \frac{\partial f_1}{\partial x_n}\left(\mathbf{x}^{(k-1)}\right) \\ \frac{\partial f_2}{\partial x_1}\left(\mathbf{x}^{(k-1)}\right) & \frac{\partial f_2}{\partial x_2}\left(\mathbf{x}^{(k-1)}\right) & \cdots & \frac{\partial f_2}{\partial x_n}\left(\mathbf{x}^{(k-1)}\right) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}\left(\mathbf{x}^{(k-1)}\right) & \frac{\partial f_n}{\partial x_2}\left(\mathbf{x}^{(k-1)}\right) & \cdots & \frac{\partial f_n}{\partial x_n}\left(\mathbf{x}^{(k-1)}\right) \end{bmatrix}$$

$$\begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ \vdots \\ y_n^{(k-1)} \end{bmatrix}$$

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ \vdots \\ y_n^{(k-1)} \end{bmatrix} = -\left(J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right)^{-1} \mathbf{F}\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)$$

<u>Example 4</u>: The nonlinear system:

$$3x_1 - cos(x_2 x_3) - \frac{1}{2} = 0$$

$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

Apply Newton's method to this problem with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

Solution: Define

$$\mathbf{F}(\mathbf{x}) = \left(f_1(x_1, x_2, x_3),\ f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)\right)^t$$

Where:

$$f_1(x_1, x_2, x_3) = 3x_1 - cos(x_2 x_3) - \frac{1}{2}$$

$$f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06$$

$$f_3(x_1, x_2, x_3) = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3}$$

The Jacobian $J(\mathbf{x})$ matrix for this system is:

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_3 \sin x_2 x_3 & x_2 \sin x_2 x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}$$

$$\mathbf{F}\left(\mathbf{x}^{(0)}\right) = (-0.199995, -2.269833417, 8.462025346)^t$$

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & 9.999833334 \times 10^{-4} & 9.999833334 \times 10^{-4} \\ 0.2 & -32.4 & 0.9950041653 \\ -0.09900498337 & -0.09900498337 & 20 \end{bmatrix}$$

Now we solve the linear system

$$J\left(\mathbf{x}^{(0)}\right)\boldsymbol{y}^{(0)} = -F\left(\mathbf{x}^{(0)}\right)$$

$$\boldsymbol{y}^{(0)} = -J\left(\mathbf{x}^{(0)}\right)^{-1} F\left(\mathbf{x}^{(0)}\right)$$

$$\boldsymbol{y}^{(0)} = \begin{bmatrix} 0.3998696728 \\ -0.08053315147 \\ -0.4215204718 \end{bmatrix}$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \boldsymbol{y}^{(0)} = \begin{bmatrix} 0.4998696782 \\ 0.01946684853 \\ -0.5215204718 \end{bmatrix}$$

Continuing for k =2, 3, …, we have:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\Vert\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\Vert_\infty$
0	**0.10000000**	**0.10000000**	**−0.10000000**	
1	**0.4998696728**	**0.0194668485**	**−0.5215204718**	**0.4215204718**
2	**0.5000142403**	**0.0015885914**	**−0.5235569638**	$\mathbf{1.788 \times 10^{-2}}$
3	**0.5000000113**	**0.0000124448**	**−0.5235984500**	$\mathbf{1.576 \times 10^{-4}}$
4	**0.50000000**	$\mathbf{8.516 \times 10^{-10}}$	**−0.5235987755**	$\mathbf{1.244 \times 10^{-5}}$
5	**0.50000000**	$\mathbf{-1.375 \times 10^{-11}}$	**−0.5235987756**	$\mathbf{8.654 \times 10^{-10}}$