The general 2^{nd} order linear PDE

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where the coefficients A, B, C, D, E, F and G are in general functions of the independent variables x, y, but do not depend on the unknown function u.

In this lecture we discuss only four cases as follows:

(1) When $B = C = E = F = 0$ the PDE becomes $Au_{xx} + Du_x = G$
(2) When $A = C = E = F = 0$ the PDE becomes $Bu_{xy} + Du_x = G$
(3) When $A = C = D = F = 0$ the PDE becomes $Bu_{xy} + Eu_y = G$
(4) When $A = B = D = F = 0$ the PDE becomes $Cu_{yy} + Eu_y = G$

We can solve the PDE by assuming that $v = u_x$ in cases (1)& (2) and $v = u_y$ in cases (3)& (4). Then the PDE becomes linear ODE. The following examples explain how to solve the PDE.

Example 1: Solve $u_{xx} - yu_x = 2y$ with $u(1,y) = -2$ and $u_x(0,y) = y - 2$

Solution : Put $u_x = v \Rightarrow u_{xx} = \frac{dv}{dx}$

So $\frac{dv}{dx} - yv = 2y$

Which is a linear ODE with $P(x) = -y$ and $Q(x) = 2y$

$I.F = e^{\int P(x)\,dx} = e^{\int -y\,dx} = e^{-yx}$ So, the integrating factor is

Then $v = \frac{1}{I.F}\int I.F\ Q(x)dx \Rightarrow v = \frac{1}{e^{-yx}}\int e^{-yx}\ 2y\,dx$

$v = e^{yx}\left(-2e^{-yx} + f(y)\right) \Rightarrow v = -2 + e^{yx}f(y)$

$u_x = v \Rightarrow u_x = -2 + e^{yx}f(y)$

$u_x(0,y) = y - 2 \Rightarrow y - 2 = -2 + f(y) \Rightarrow f(y) = y$

$u_x = -2 + ye^{yx}$

$u(x,y) = \int(-2 + ye^{yx})dx = -2x + e^{yx} + g(y)$

$u(1,y) = -2 \Rightarrow -2 = -2 + e^y + g(y) \Rightarrow g(y) = e^y$

Then $u(x,y) = -2x + e^{yx} + e^y$

Example 2: Solve $u_{xy} + u_x = 1$ with $u(0,y) = 0$ and $u_x(x,0) = \sin x$□

Solution : Put $u_x = v \Rightarrow u_{xy} = \dfrac{dv}{dy} \Rightarrow \dfrac{dv}{dy} + v = 1$

Which is a linear ODE with $P(y) = 1$ and $Q(y) = 1$

$I.F = e^{\int P(y)dy} = e^{\int dy} = e^y$ So, the integrating factor is

Then $v = \dfrac{1}{I.F}\int I.F\ Q(y)dy \Rightarrow v = e^{-y}\int e^y\ dy$

$$v = e^{-y}\left(e^y + f'(x)\right) \Rightarrow v = 1 + e^{-y}f(x) \Rightarrow u_x = 1 + e^{-y}f(x)$$

$$u_x(x,0) = \sin x \Rightarrow \sin x = 1 + f(x) \Rightarrow f(x) = -1 + \sin x$$

Then $u(x,y) = \int \left(1 + e^{-y}(-1 + \sin x)\right)dx = \int (1 - e^{-y} + e^{-y}\sin x)\,dx$

$$u(x,y) = x - xe^{-y} - e^{-y}\cos x + g(y)$$

$$u(0,y) = 0 \Rightarrow g(y) = e^{-y}$$

$$u(x,y) = x - xe^{-y} - e^{-y}\cos x + e^{-y}$$

Example 3: Solve $xu_{xy} + 2u_y = 9y^2x$ with $u_y(1,y) = 0$ and $u(x,0) = 1$□

Solution : Put $u_y = v \Rightarrow u_{xy} = \dfrac{dv}{dx}$

$$xu_{xy} + 2u_y = y^2 \Rightarrow x\frac{dv}{dx} + 2v = 9y^2x \Rightarrow \frac{dv}{dx} + \frac{2}{x}v = 9y^2$$

$$I.F = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$v = \frac{1}{x^2}\int 9x^2\ y^2\ dx = \frac{1}{x^2}\left(3x^3y^2 + f(y)\right) \Rightarrow u_y = 3xy^2 + \frac{1}{x^2}f(y)$$

$$u_y(1,y) = 0 \Rightarrow f(y) = -3y^2$$

$$u_y = 3xy^2 - \frac{3y^2}{x^2}$$

$$u(x,y) = \int \left(3xy^2 - \frac{3y^2}{x^2}\right)dy = xy^3 + \frac{y^3}{x^2} + g(x)$$

$$u(x,0) = 1 \Rightarrow g(x) = 1$$

$$u(x,y) = xy^3 + \frac{y^3}{x^2} + 1$$

Example 4: Solve $u_{yy} + 2u_y = 2\sin x$ with $u(x,0) = 0$ and $u_y(x,0) = 3\sin x$

Solution : Put $\quad u_y = v \Rightarrow u_{yy} = \dfrac{dv}{dy}$

$$\frac{dv}{dy} + 2v = 2\sin x$$

$$I.F = e^{\int P(y)dy} = e^{\int 2dy} = e^{2y}$$

$$v = e^{-2y}\int 2e^{2y} \sin x \, dy$$

$$u_y = e^{-2y}\left(e^{2y}\sin x + f(x)\right) = \sin x + e^{-2y} f(x)$$

$$u_y(x,0) = 3\sin x \Rightarrow 3\sin x = \sin x + f(x) \Rightarrow f(x) = 2\sin x$$

$$u_y = \sin x + 2\sin x \, e^{-2y}$$

Then $\quad u(x,y) = \displaystyle\int (\sin x + 2\sin x \, e^{-2y}) \, dy$

$$u(x,y) = y\sin x - \sin x \, e^{-2y} + g(x)$$

$$u(x,0) = 0 \Rightarrow g(x) = \sin x$$

$$u(x,y) = y\sin x - \sin x \, e^{-2y} + \sin x = \sin x\,(y - e^{-2y} + 1)$$

H.W: Solve the PDE

1. $u_{xx} + 3u_x = 3e^y$ with $u(1,y) = \frac{2}{3}e^{y-3}$ and $u_x(0,y) = 2e^y$

2. $u_{xy} - u_x = 2$ with $u(0,y) = \tan y$ and $u_x(x,0) = \sec^2 x - 2$

3. $xu_{xy} + u_y = 0$ with $u_y(1,y) = \cos y$ and $u\left(x, \frac{\pi}{2}\right) = 0$

4. $u_{yy} - u_y = \tan^{-1} x$ with $u(x,0) = 0$ and $u_y(x,0) = 0$