Sequences

A sequence of real numbers is a function $a : N \to R$.

The sequence is denoted by $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ or $\{a_n\}$

For example, the expression $\{2n\}$ denotes the sequence $\{2, 4, 6, \cdots\}$

The number a_n is called the general term of the sequence $\{a_n\}$.

Example 1: Write down the first few terms of each of the following sequences.

1. $\left\{\dfrac{n+1}{n^2}\right\}$ 2. $\left\{\dfrac{2n}{n+3}\right\}$

3. $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}$ 4. $\{c_n\}$ where $c_n = n$th digit of π

Solution

1. $\left\{\dfrac{n+1}{n^2}\right\} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \cdots\right\}$

2. $\left\{\dfrac{2n}{n+3}\right\} = \left\{\underbrace{\frac{1}{2}}_{n=1}, \underbrace{\frac{4}{5}}_{n=2}, \underbrace{1}_{n=3}, \underbrace{\frac{8}{7}}_{n=4}, \underbrace{\frac{5}{4}}_{n=5}, \cdots\right\}$

3. $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\} = \left\{\underbrace{\frac{1}{2}}_{n=1}, \underbrace{\frac{-1}{4}}_{n=2}, \underbrace{\frac{1}{8}}_{n=3}, \underbrace{\frac{-1}{16}}_{n=4}, \underbrace{\frac{1}{32}}_{n=5}, \cdots\right\}$

4. $\{c_n\} = \{3,1,4,1,5, \cdots\}$

convergent and divergent sequence:

If $\lim\limits_{n\to\infty} a_n$ exists and is finite we say that the sequence is convergent. If $\lim\limits_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence diverges. Note that sometimes we will say the sequence diverges to ∞ if $\lim\limits_{n\to\infty} a_n = \infty$ and if $\lim\limits_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence diverges to $-\infty$.

Example 2: Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{n^2+1}{(n+1)^2}\right\}$ 2. $\left\{\dfrac{n+17}{\sqrt{2n^2+3n}}\right\}$ 3. $\left\{\dfrac{e^n}{n^3}\right\}$

4. $\left\{\sqrt{n+4}-\sqrt{n}\right\}$ 5. $\left\{\sqrt{n^2+5n}-n\right\}$

Solution

1. $$\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2}=\lim_{n\to\infty}\frac{2n}{2(n+1)}$$
$$=\lim_{n\to\infty}\frac{2}{2}=1 \quad \text{converges to } 1$$

2. $$\lim_{n\to\infty}\frac{n+17}{\sqrt{2n^2+3n}}=\lim_{n\to\infty}\frac{1+\frac{17}{n}}{\sqrt{2+\frac{3}{n}}}$$
$$=\frac{1}{\sqrt{2}} \quad \text{converges to } \frac{1}{\sqrt{2}}$$

3. $$\lim_{n\to\infty}\frac{e^n}{n^3}=\lim_{n\to\infty}\frac{e^n}{3n^2}=\lim_{n\to\infty}\frac{e^n}{6n}$$
$$=\lim_{n\to\infty}\frac{e^n}{6}=\infty \quad \text{diverges}$$

4. $$\lim_{n\to\infty}\left(\sqrt{n+4}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\sqrt{n+4}-\sqrt{n}\right)\times\frac{\left(\sqrt{n+4}+\sqrt{n}\right)}{\left(\sqrt{n+4}+\sqrt{n}\right)}$$
$$=\lim_{n\to\infty}\frac{n+4-n}{\left(\sqrt{n+4}+\sqrt{n}\right)}=0 \quad \text{converges to } 0$$

5. $$\lim_{n\to\infty}\left(\sqrt{n^2+5n}-n\right)\times\frac{\sqrt{n^2+5n}+n}{\sqrt{n^2+5n}+n}=\lim_{n\to\infty}\frac{n^2+5n-n^2}{\sqrt{n^2+5n}+n}$$
$$=\lim_{n\to\infty}\frac{5n}{\sqrt{n^2+5n}+n}=\lim_{n\to\infty}\frac{5}{\sqrt{1+\frac{5}{n}}+1}$$
$$=\frac{5}{2} \quad \text{converges to } \frac{5}{2}$$

Limits that arise frequently

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \;\; ; \;\; (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \quad \forall \;\; |x| < 1$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Calculation of the limits

1. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

2. Let $a = \lim_{n\to\infty} \sqrt[n]{n}$ then $\ln a = \ln \lim_{n\to\infty} \sqrt[n]{n}$

$$\ln a = \ln \lim_{n\to\infty} (n)^{1/n}$$

$$\ln a = \lim_{n\to\infty} \ln(n)^{1/n}$$

$$\ln a = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$a = e^0 = 1$$

$$\text{So } \lim_{n\to\infty} \sqrt[n]{n} = 1$$

5. Let $a = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ then $\ln a = \lim_{n\to\infty} \ln \left(1 + \frac{x}{n}\right)^n$

$$\ln a = \lim_{n\to\infty} n \ln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{\frac{1}{n}}$$

$$\ln a = \lim_{n\to\infty} \frac{\frac{1}{\left(1 + \frac{x}{n}\right)} \cdot \frac{-x}{n^2}}{\frac{-1}{n^2}} = x$$

$$a = e^x$$

$$\text{So } \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Example 3: Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{1+\ln n}{n}\right\}$ 2. $\left\{\dfrac{1}{(0.6)^n}\right\}$ 3. $\left\{\left(\dfrac{2^n}{n}\right)^{1/n}\right\}$

4. $\left\{\sqrt[n]{n^2+n}\right\}$ 5. $\left\{\ln\left(\dfrac{n-2}{n}\right)^n\right\}$ 6. $\left\{\dfrac{n!}{10^n}\right\}$

Solution

1. $\lim\limits_{n\to\infty}\dfrac{1+\ln n}{n}=\lim\limits_{n\to\infty}\dfrac{1}{n}+\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=\ 0$ converges to 0

2. $\lim\limits_{n\to\infty}\dfrac{1}{(0.6)^n}=\dfrac{1}{\lim\limits_{n\to\infty}(0.6)^n}=\dfrac{1}{0}=\infty$ diverges

3. $\lim\limits_{n\to\infty}\left(\dfrac{2^n}{n}\right)^{1/n}=\lim\limits_{n\to\infty}\dfrac{2}{\sqrt[n]{n}}=2$ converges to 2

4. $\lim\limits_{n\to\infty}\sqrt[n]{n^2+n}=\lim\limits_{n\to\infty}\sqrt[n]{n(n+1)}=\lim\limits_{n\to\infty}\sqrt[n]{n}\times\lim\limits_{n\to\infty}\sqrt[n]{n+1}=1$ converges to1

5. $\lim\limits_{n\to\infty}\ln\left(\dfrac{n-2}{n}\right)^n=\ln\lim\limits_{n\to\infty}\left(1+\dfrac{-2}{n}\right)^n=\ln e^{-2}=-2$ converges to -2

6. $\lim\limits_{n\to\infty}\dfrac{n!}{10^n}=\dfrac{1}{\lim\limits_{n\to\infty}\dfrac{10^n}{n!}}=\dfrac{1}{0}=\infty$ diverges

Exercises

Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{3+\ln n^n}{n^2}\right\}$ 2. $\left\{\left(\dfrac{3}{n}\right)^{1/n}\right\}$ 3. $\left\{\left(\dfrac{n+5}{n}\right)^n\right\}$

4. $\left\{\sqrt[n]{4^n n}\right\}$ 5. $\left\{\dfrac{n^5+2n}{3n^4+n^2}\right\}$ 6. $\left\{\dfrac{8^{n+1}}{n!}\right\}$

7. $\left\{\dfrac{e^n+n^2}{e^n-2n^2}\right\}$ 8. $\left\{\sqrt{n(n+2)}-n\right\}$ 9. $\left\{\left(\dfrac{3^n}{2n+1}\right)^{1/n}\right\}$

10. $\left\{\dfrac{3n^2-\ln n}{n^2+3n^{3/2}}\right\}$ 11. $\left\{\left(\dfrac{n-3}{n}\right)^n\right\}$