Definition 1 **Joint cumulative distribution function**

Let $X_1, X_2 \dots, X_k$ be k random variables all defined on the same probability space $(\Omega, \mathcal{A}, P)$,or $(X_1, X_2 \dots, X_k)$ is k-dimensional random variable. The joint cumulative distribution function of $X_1, X_2 \dots, X_k$

denoted by $F(x_1, \dots, x_k)$, is defined as:

$$F_{X_1,X_2\dots,X_k}(.,\dots,.) = P[X_1 \le x_1, \dots, X_k \le x_k] \text{ for all } (x_1, \dots, x_k) \in \mathbb{R}^k$$

Remark

1) If $X_1, X_2 \dots, X_k$ be k discrete random variables, then the joint probability mass function j.p.m.f. of $(X_1, X_2 \dots, X_k)$, denoted by $f_{X_1,X_2\dots,X_k}(.,\dots,.)$ is defined to be

$$f_{X_1,X_2\dots,X_k}(x_1,\dots,x_k) = f(x_1,\dots,x_k) = P(x_1,\dots,x_k)$$

$$f(x_1,\dots,x_k) = \begin{cases} P[X_1 = x_1, \dots, X_k = x_k], & (x_1, \dots, x_k) \in \mathbb{R}^k \\ 0, & otherwise \end{cases}$$

That has the following properties:

a) $f(x_1,\dots,x_k) \ge 0 \ \forall (x_1, \dots, x_k) \in \mathbb{R}^k$,
b) $\sum_{\forall(x_1,\dots,x_k)} f(x_1,\dots,x_k) = 1$.
   For example if $k = 2$, then the joint probability mass function of (X_1, X_2) is $f(x_1, x_2) = P[X_1 = x_1, X_2 = x_2]$ and the joint cumulative distribution function of (X_1, X_2) is
   $F_{X_1,X_2}(x_1, x_2) = P[X_1 \le x_1, X_2 \le x_2]$.

2) If $X_1, X_2 \dots, X_k$ be k continuous random variables, or $(X_1, X_2 \dots, X_k)$ is k-dimensional continuous random variable, then the joint probability density function of $(X_1, X_2 \dots, X_k)$, denoted by $f_{X_1,X_2\dots,X_k}(.,\dots,.) = f(x_1,\dots,x_k)$. Therefore $F_{X_1,X_2\dots,X_k}(.,\dots,.) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \dots \int_{-\infty}^{x_k} f(u_1,\dots,u_k) du_1 \dots du_k$ for all $(x_1, \dots, x_k) \in \mathbb{R}^k$

The joint probability density function of $(X_1, X_2 \dots, X_k)$ has the following properties:

a) $f(x_1, \dots, x_k) \geq 0 \;\; \forall (x_1, \dots, x_k) \in \mathbb{R}^k$,
b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_k) dx_1 \dots dx_k = 1$.

For example

If $k = 2$, then the joint probability density function j.p.d.f. of (X_1, X_2) is $f(x_1, x_2)$ and the joint cumulative distribution function of (X_1, X_2) is $F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u_1, u_2) du_1 du_2$.

If $R = \{(X_1, X_2): \; a_1 < X_1 \leq b_1; \; a_2 < X_2 \leq b_2\}$,then

$$P[a_1 < X_1 \leq b_1, a_2 < X_2 \leq b] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) dx_1 dx_2$$

EXAMPLE 1 Consider the bivariate function

$$f(x, y) = K(x + y) I_{(0,1)}(x) I_{(0,1)}(y) = K(x + y) I_U(x, y),$$

where $U = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 1\}$, a unit square. Can the constant K be selected so that $f(x, y)$ will be a joint probability density function? If K is positive, $f(x, y) \geq 0$.

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Kf(x, y)\, dx\, dy &= \int_0^1 \int_0^1 K(x + y)\, dx\, dy \\ &= K \int_0^1 \int_0^1 (x + y)\, dx\, dy \\ &= K \int_0^1 (\tfrac{1}{2} + y)\, dy \\ &= K(\tfrac{1}{2} + \tfrac{1}{2}) \\ &= 1 \end{aligned}$$

Properties of bivariate cumulative distribution function $F(\cdot, \cdot)$

(i) $F(-\infty, y) = \lim_{x \to -\infty} F(x, y) = 0$ for all y, $F(x, -\infty) = \lim_{y \to -\infty} F(x, y) = 0$ for all x, and $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = F(\infty, \infty) = 1$.

(ii) If $x_1 < x_2$ and $y_1 < y_2$, then $P[x_1 < X \le x_2; y_1 < Y \le y_2] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \ge 0$.

(iii) $F(x, y)$ is right continuous in each argument; that is, $\lim_{0 < h \to 0} F(x + h, y) = \lim_{0 < h \to 0} F(x, y + h) = F(x, y)$.

This joint cumulative distribution function of (X_1, X_2) , $F_{X_1,X_2}(.,.)$is called bivariate cumulative distribution function.

Definition 2 **Marginal cumulative distribution function** If $F_{X,Y}(\cdot, \cdot)$ is the joint cumulative distribution function of X and Y, then the cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ are called *marginal cumulative distribution functions.* ////

Remark $F_X(x) = F_{X,Y}(x, \infty)$, and $F_Y(y) = F_{X,Y}(\infty, y)$; that is, knowledge of the joint cumulative distribution function of X and Y implies knowledge of the two marginal cumulative distribution functions. ////

That is $F_{X_1}(x_1) = F_{X_1,X_2}(x_1, \infty) = \int_{-\infty}^{x_1} \int_{a_2}^{\infty} f(u_1, x_2) dx_2 du_1$

<u>Marginal probability density(mass) functions</u>

Let $X_1, X_2 \dots , X_k$ be k c.r.v.'s (d.r.v.'s) that have j.p.d.f. $f_{X_1,X_2\dots,X_k}(., \dots, .)$, then the marginal of any subset of the jointly continuous random variables $X_1, X_2 \dots , X_k$ as $X_{i1}, X_{i2} \dots , X_{im}$ can be defined as

$$\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots . \int_{-\infty}^{x_k} f(x_1, \dots, x_{i1}, \dots, x_{im}, \dots, x_k) dx_1 \dots . dx_k = f(x_{i1}, \dots, x_{im})$$

For example if $X_1, X_2 \dots , X_5$ c.r.v.'s , then the marginal of X_1, X_2 is defined as

$$\int_{-\infty}^{x_3}\int_{-\infty}^{x_4}\int_{-\infty}^{x_5} f(x_1,x_2,x_3,x_4,x_5)dx_3dx_4dx_5 = f(x_1,x_2)$$

Therefore the marginal of X_i is

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\cdots\int_{-\infty}^{x_5} f\left(x_1,x_2,x_{i,}\ldots.,x_5\right)dx_1dx_2\ldots..dx_5 = f(x_i)$$

Let $i=3$, then the marginal of X_3 is

$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_4}\int_{-\infty}^{x_5} f(x_1,x_2,x_3,x_4,x_5)dx_1dx_2dx_4dx_5 = f(x_3)$

Similarly if we have d.r.v.'s , that is

$$P(x_1,x_2) = f(x_1,x_2) = \sum_{\forall x_3}\sum_{\forall x_4}\sum_{\forall x_5} f(x_1,x_2,x_3,x_4,x_5)$$

and

$$P(x_1) = f(x_1,x_2) = \sum_{\forall x_2}\sum_{\forall x_3}\sum_{\forall x_4}\sum_{\forall x_5} f(x_1,x_2,x_3,x_4,x_5)$$

Example 1. Let the joint p.d.f. of X_1 and X_2 be

$$f(x_1, x_2) = \frac{x_1 + x_2}{21}, \qquad x_1 = 1, 2, 3, x_2 = 1, 2,$$

$$= 0 \text{ elsewhere.}$$

Then, for instance,

$$\Pr(X_1 = 3) = f(3, 1) + f(3, 2) = \tfrac{3}{7}$$

and

$$\Pr(X_2 = 2) = f(1, 2) + f(2, 2) + f(3, 2) = \tfrac{4}{7}.$$

On the other hand, the marginal p.d.f. of X_1 is

$$f_1(x_1) = \sum_{x_2=1}^{2} \frac{x_1 + x_2}{21} = \frac{2x_1 + 3}{21}, \qquad x_1 = 1, 2, 3,$$

zero elsewhere, and the marginal p.d.f. of X_2 is

$$f_2(x_2) = \sum_{x_1=1}^{3} \frac{x_1 + x_2}{21} = \frac{6 + 3x_2}{21}, \qquad x_2 = 1, 2,$$

zero elsewhere. Thus the preceding probabilities may be computed as $\Pr(X_1 = 3) = f_1(3) = \frac{3}{7}$ and $\Pr(X_2 = 2) = f_2(2) = \frac{4}{7}$.

Example 2. Let X_1 and X_2 have the joint p.d.f.

$$f(x_1, x_2) = 2, \qquad 0 < x_1 < x_2 < 1,$$
$$= 0 \text{ elsewhere.}$$

Then the marginal probability density functions are, respectively,

$$f_1(x_1) = \int_{x_1}^{1} 2\, dx_2 = 2(1 - x_1), \qquad 0 < x_1 < 1,$$
$$= 0 \text{ elsewhere,}$$

and

$$f_2(x_2) = \int_0^{x_2} 2\, dx_1 = 2x_2, \qquad 0 < x_2 < 1,$$
$$= 0 \text{ elsewhere.}$$

The conditional p.d.f. of X_1, given $X_2 = x_2$, is

$$f(x_1|x_2) = \frac{2}{2x_2} = \frac{1}{x_2}, \qquad 0 < x_1 < x_2,\ 0 < x_2 < 1,$$
$$= 0 \text{ elsewhere.}$$

EXAMPLE Consider the joint probability density

$$f_{X,Y}(x, y) = (x + y)I_{(0,1)}(x)I_{(0,1)}(y).$$

$$\begin{aligned}F_{X,Y}(x, y) &= I_{(0,1)}(x)I_{(0,1)}(y) \int_0^y \int_0^x (u + v)\, du\, dv \\ &\quad + I_{(0,1)}(x)I_{[1,\infty)}(y) \int_0^1 \int_0^x (u + v)\, du\, dv \\ &\quad + I_{[1,\infty)}(x)I_{(0,1)}(y) \int_0^y \int_0^1 (u + v)\, du\, dv \\ &\quad + I_{[1,\infty)}(x)I_{[1,\infty)}(y) \\ &= \tfrac{1}{2}\{(x^2y + xy^2)I_{(0,1)}(x)I_{(0,1)}(y) + (x^2 + x)I_{(0,1)}(x)I_{[1,\infty)}(y) \\ &\quad + (y + y^2)I_{[1,\infty)}(x)I_{(0,1)}(y)\} + I_{[1,\infty)}(x)I_{[1,\infty)}(y).\end{aligned}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

$$= I_{(0,1)}(x) \int_0^1 (x + y)\, dy$$

$$= (x + \tfrac{1}{2}) I_{(0,1)}(x);$$

or,

$$f_X(x) = \frac{\partial F_{X,Y}(x, \infty)}{\partial x}$$

$$= \frac{\partial F_X(x)}{\partial x}$$

$$= I_{(0,1)}(x) \frac{\partial}{\partial x}\left(\frac{x^2 + x}{2}\right)$$

$$= (x + \tfrac{1}{2}) I_{(0,1)}(x).$$

"Introduction to the Theory qf Statistics" By "Alexander McFar1ane Mood"