DISCRETE DISTRIBUTIONS

Discrete Uniform Distribution

Definition 1 Discrete uniform distribution Each member of the family of discrete density functions

$$f(x) = f(x; N) = \begin{cases} \frac{1}{N} & \text{for } x = 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases} = \frac{1}{N} I_{\{1, 2, \ldots, N\}}(x), \qquad (1)$$

where the parameter N ranges over the positive integers, is defined to have a *discrete uniform distribution*. A random variable X having a density given in Eq. (1) is called a *discrete uniform random variable*. ////

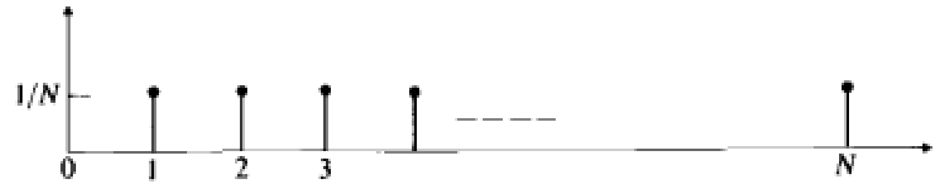


FIGURE 1

Density of discrete uniform

Theorem 1 If X has a discrete uniform distribution, then $\mathscr{E}[X] = (N+1)/2$,

$$\text{var}[X] = \frac{(N^2 - 1)}{12}, \text{ and } m_X(t) = \mathscr{E}[e^{tX}] = \sum_{j=1}^{N} e^{jt} \frac{1}{N}.$$

PROOF

$$\mathscr{E}[X] = \sum_{j=1}^{N} j \frac{1}{N} = \frac{N+1}{2}.$$

$$\text{var}[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = \sum_{j=1}^{N} j^2 \frac{1}{N} - \left(\frac{N+1}{2}\right)^2$$

$$= \frac{N(N+1)(2N+1)}{6N} - \frac{(N+1)^2}{4} = \frac{(N+1)(N-1)}{12}.$$

$$\mathscr{E}[e^{tX}] = \sum_{j=1}^{N} e^{jt} \frac{1}{N}.$$

Remark The discrete uniform distribution is sometimes defined in density form as $f(x;\ N) = [1/(N+1)]I_{\{0,1,\ldots,N\}}(x)$, for N a nonnegative integer. If such is the case, the formulas for the mean and variance have to be modified accordingly. ////

Bernoulli and Binomial Distributions

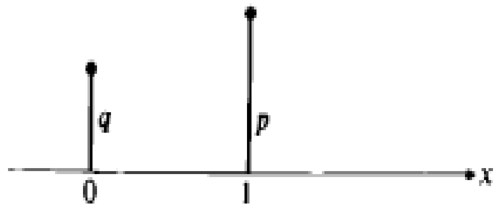
Definition 2 **Bernoulli distribution** A random variable X is defined to have a *Bernoulli distribution* if the discrete density function of X is given by

$$f_X(x) = f_X(x;p)$$
$$= \begin{cases} p^x(1-p)^{1-x} & \text{for } x = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases} = p^x(1-p)^{1-x}I_{\{0,1\}}(x), \qquad (2)$$

where the parameter p satisfies $0 \le p \le 1$. $1-p$ is often denoted by q.

FIGURE 2

Bemou1li density



Theorem 2 If X has a Bernoulli distribution, then

$$\mathscr{E}[X] = p, \qquad \text{var}\,[X] = pq, \qquad \text{and} \qquad m_X(t) = pe^t + q. \qquad (3)$$

PROOF $\mathscr{E}[X] = 0 \cdot q + 1 \cdot p = p.$

$$\text{var}\,[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = 0^2 \cdot q + 1^2 \cdot p - p^2 = pq.$$
$$m_X(t) = \mathscr{E}[e^{tX}] = q + pe^t.$$ ////

EXAMPLE 2 For a given arbitrary probability space $(\Omega, \mathscr{A}, P[\cdot])$ and for A belonging to $\mathscr{A}$, define the random variable X to be the indicator function of A; that is, $X(\omega) = I_A(\omega)$; then X has a Bernoulli distribution with parameter $p = P[X = 1] = P[A]$. ////

Definition 3 Binomial distribution A random variable X is defined to have a *binomial distribution* if the discrete density function of X is given by

$$f_X(x) = f_X(x; n, p) = \begin{cases} \binom{n}{x} p^x q^{n-x} & \text{for } x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$= \binom{n}{x} p^x q^{n-x} I_{\{0, 1, \ldots, n\}}(x),$$

where the two parameters n and p satisfy $0 \leq p \leq 1$, n ranges over the positive integers, and $q = 1 - p$. A distribution defined by the density function given in Eq. (4) is called a *binomial distribution*. ////

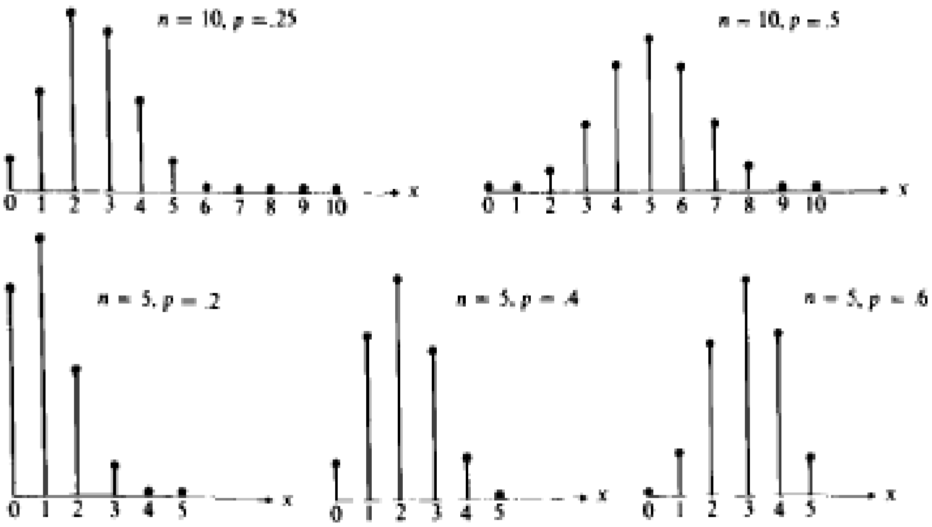


FIGURE 3

Binomial densities

Theorem 3 If X has a binomial distribution, then

$$\mathscr{E}[X] = np, \qquad \text{var}[X] = npq, \qquad \text{and} \qquad m_X(t) = (q + pe^t)^n. \tag{5}$$

PROOF

$$m_X(t) = \mathscr{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$
$$= (pe^t + q)^n.$$

EXAMPLE 4 Consider sampling with replacement from an urn containing M balls, K of which are defective. Let X represent the number of defective balls in a sample of size n. The individual draws are Bernoulli trials where "defective" corresponds to "success," and the experiment of taking a sample of size n with replacement consists of n repeated independent Bernoulli trials where $p = P[\text{success}] = K/M$; so X has the binomial distribution

$$\binom{n}{x}\left[\frac{K}{M}\right]^{x}\left[1-\frac{K}{M}\right]^{n-x} \qquad \text{for} \quad x = 0, 1, \ldots, n, \tag{6}$$