University of Babylon
College of Engineering
Department of Environmental Engineering
Engineering Analysis I (ENAN 103)

Numerical Integration

Equal Segments Trapezoidal Method

Undergraduate Level, 3th Stage

Mr. Waleed Ali Tameemi
Engineer/ College of Engineering/ Babylon University
M.Sc. Civil Engineering/ the University of Kansas/ USA

2016-2017

Lecture Outline

1.0 Introduction

2.0 Trapezoidal Method

2.1 Equal Segments Trapezoidal Method

2.2 Unequal Segments Trapezoidal Method

3.0 Simpson's Rule Method

3.1 Simpson's 1/3 Rule

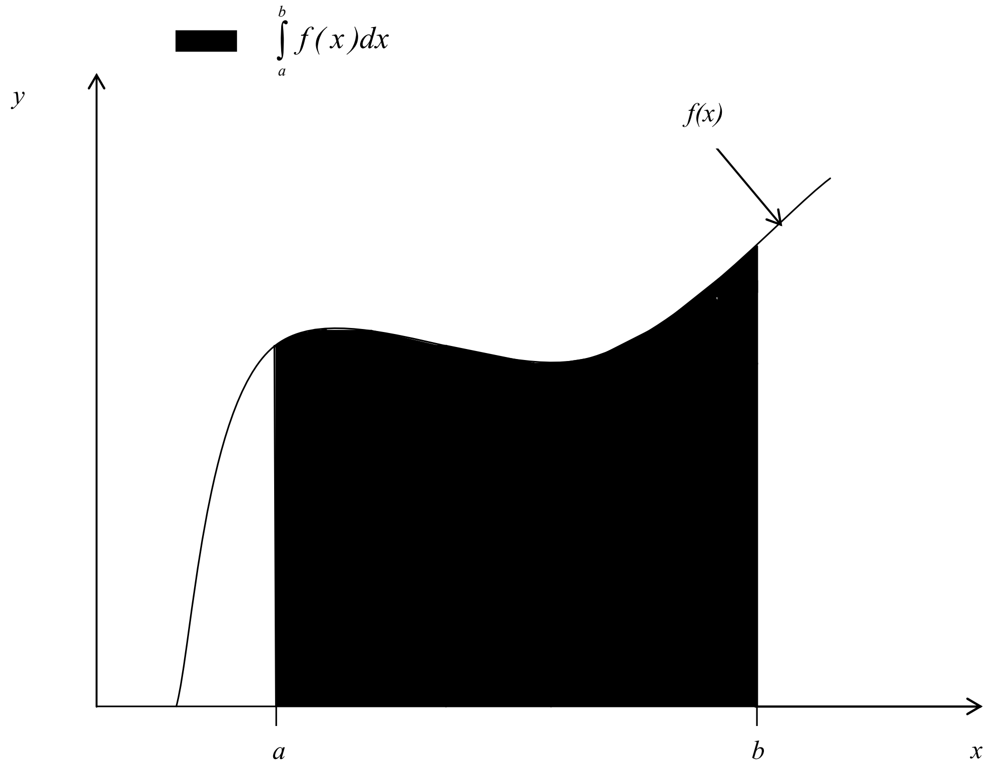
3.2 Simpson's 3/8 Rule

4.0 Gaussian Quadrature Method

5.0 Summary

1.0 – Introduction

Integration can be defined as the area under the curve of a function plotted on a graph. The following sections will discuss numerical methods of integration.



$$I = \int_a^b f(x)dx$$

where:

I: the Integration value,

$f(x)$: the integrand function,

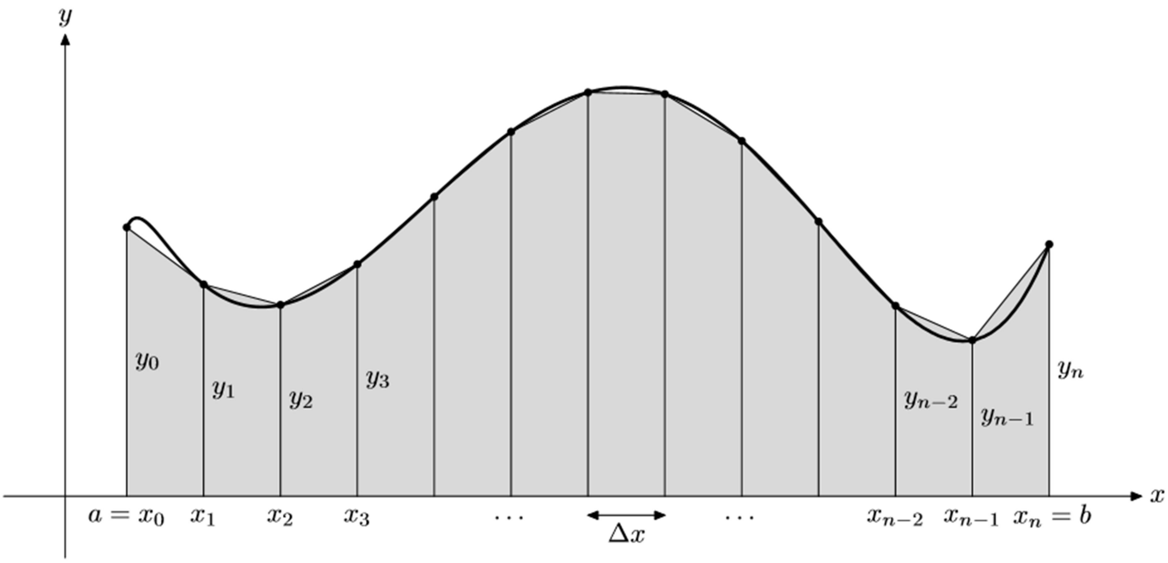
a: lower limit of integration,

b: upper limit of integration.

2.0 – Trapezoidal Method

2.1 – Equal Segments Trapezoidal Method

The integration of a function can be approximately calculated using the trapezoid method by estimating the area under the curve of the given function and as follows:



$$\int_a^b f(x)dx \quad \Delta x\frac{y_0 + y_1}{2} + \Delta x\frac{y_1 + y_2}{2} + \cdots + \Delta x\frac{y_{n-1} + y_n}{2}$$

$$\boldsymbol{\int_a^b f(x)dx = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i]}$$

$$\Delta x = \frac{b-a}{n}$$

$$\epsilon = \left|\frac{True\ Value - Appro \qquad Value}{True\ Value}\right| \times 100\%$$

where:

n: the number of segments (the higher the n, the more accurate solution),

∈: the error value.

$x_0 = a$ $\qquad\qquad$ $y_0 = f(x_0)$

$x_1 = x_0 + \Delta x$ $\qquad\qquad$ $y_1 = f(x_1)$

$x_n = x_{n-1} + \Delta x = b$ $\qquad\qquad$ $y_n = f(x_n)$

Ex1: Calculate the value of the given integration.

$$\int_{8}^{30} \{2000 \ln[\frac{140000}{140000 - 2100x}] - 9.8x\}dx$$

The number of segments (n) is equal to 2,4, and 8.

Compare your solution with the exact solution ($I = 11061$).

Solution:

$a = 8 \quad , \quad b = 30$

First, two segments (n=2)

$$\Delta x = \frac{b-a}{n} = \frac{30-8}{2} = 11$$

$$y_i = f(x_i) = 2000 \ln[\frac{140000}{140000 - 2100x}] - 9.8x_i$$

i	x	y
0	$x_0 = a = 8$	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (8)}] - 9.8 \times (8) = 177.27$

1	8+11=19	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (19)}] - 9.8 \times (19) = 484.8$
2	19+11=30=b	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (30)}] - 9.8 \times (30) = 901.7$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i] = \frac{11}{2}[177.27 + 901.67 + 2 \times 484.75] = 11266.4$$

Compare with the true value:

$$\in = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{11061 - 11266.4}{11061}\right| \times 100\%$$

$$= 1.86\%$$

Second, four segments (n=4)

$$\Delta x = \frac{b-a}{n} = \frac{30-8}{4} = 5.5$$

$$y_i = f(x_i) = 2000 \ln[\frac{140000}{140000 - 2100x}] - 9.8x_i$$

i	x	y
0	$x_0 = a = 8$	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (8)}] - 9.8 \times (8) = 177.3$
1	8+5.5=13.5	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (13.5)}] - 9.8 \times (13.5) = 320.2$

2	13.5+5.5=19	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (19)}] - 9.8 \times (19) = 484.7$
3	19+5.5=24.5	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (24.5)}] - 9.8 \times (24.5) = 676.1$
4	24.4+5.5=30=b	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (30)}] - 9.8 \times (30) = 901.7$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i] = \frac{5.5}{2}[177.3 + 901.7 + 2 \times (320.2 + 484.75 + 676.1)] = 11113$$

Compare with the true value:

$$\epsilon = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{11061 - 11113}{11061}\right| \times 100\% = 0.47\%$$

Third, eight segments (n=8)

$$\Delta x = \frac{b-a}{n} = \frac{30-8}{8} = 2.75$$

$$y_i = f(x_i) = 2000 \ln[\frac{140000}{140000 - 2100x}] - 9.8x_i$$

i	x	y
0	$x_0 = a$ =8	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (8)}] - 9.8 \times (8) = 177.3$
1	8+2.75=10.75	$2000 \times \ln[\frac{140000}{140000 - 2100 \times (10.75)}] - 9.8 \times (10.75) = 246.3$

2	10.75+2.75=13.5	$2000 \times \ln[\frac{140000}{140000-2100\times(13.5)}] - 9.8 \times (13.5) = 320.2$
3	13.5+2.75=16.25	$2000 \times \ln[\frac{140000}{140000-2100\times(16.25)}] - 9.8 \times (16.25) = 399.5$
4	16.25+2.75=19	$2000 \times \ln[\frac{140000}{140000-2100\times(19)}] - 9.8 \times (19) = 484.7$
5	19+2.75=21.75	$2000 \times \ln[\frac{140000}{140000-2100\times(21.75)}] - 9.8 \times (21.75) = 576.6$
6	21.75+2.75=24.5	$2000 \times \ln[\frac{140000}{140000-2100\times(24.5)}] - 9.8 \times (24.5) = 676.1$
7	24.5+2.75=27.25	$2000 \times \ln[\frac{140000}{140000-2100\times(27.25)}] - 9.8 \times (27.25) = 784$
8	27.25+2.75=30=b	$2000 \times \ln[\frac{140000}{140000-2100\times(30)}] - 9.8 \times (30) = 901.7$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i]$$
$$= \frac{2.75}{2}[177.3 + 901.7 + 2 \times (246.3 + 320.2 + 399.5 + 484.75 + 576.6 + 676.1 + 784)] = 11074$$

Compare with the true value:

$$\in = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{11061 - 11074}{11061}\right| \times 100\% = 0.118\%$$

Ex2: Calculate the value of the given integration.

$$\int_{0}^{10} \frac{300x}{1+e^{x}}\,\mathrm{dx}$$

The number of segments (n) is equal to 2,4, and 8.

Compare your solution with the exact solution (I $= 246.59$).

Solution:

$a = 0 \quad , \quad b = 10$

First, two segments (n=2)

$$\Delta x = \frac{b-a}{n} = \frac{10-0}{2} = 5$$

$$y_i = f(x_i) = \int_{0}^{10} \frac{300x}{1+e^{x}}\,\mathrm{dx}$$

i	x	y
0	$x_0 = a =0$	$\frac{300\times(0)}{1+e^{(0)}} = 0$
1	0+5=5	$\frac{300\times(5)}{1+e^{(5)}} = 10.04$
2	5+5=10=b	$\frac{300\times(10)}{1+e^{(10)}} = 0.14$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i] = \frac{5}{2}[0 + 0.14 + 2 \times 10.04] = 50.55$$

Compare with the true value:

$$\in = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{246.59 - 50.55}{246.59}\right| \times 100\% = 79.5\%$$

Second, four segments (n=4)

$$\Delta x = \frac{b-a}{n} = \frac{10-0}{4} = 2.5$$

$$y_i = f(x_i) = \int_0^{10} \frac{300x}{1+e^x}\,\mathrm{dx}$$

i	x	y
0	$x_0 = a = 0$	$\frac{300 \times (0)}{1+e^{(0)}} = 0$
1	0+2.5=2.5	$\frac{300 \times (2.5)}{1+e^{(2.5)}} = 56.89$
2	2.5+2.5=5	$\frac{300 \times (5)}{1+e^{(5)}} = 10.04$
3	5+2.5=7.5	$\frac{300 \times (7.5)}{1+e^{(7.5)}} = 1.24$
4	7.5+2.5=10=b	$\frac{300 \times (10)}{1+e^{(10)}} = 0.14$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i] = \frac{2.5}{2}[0 + 0.14 + 2 \times (56.89 + 10.04 + 1.24)] = 170.6$$

Compare with the true value:

$$\in = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{246.59 - 170.6}{246.59}\right| \times 100\% = 30.82\%$$

Third, eight segments (n=8)

$$\Delta x = \frac{b-a}{n} = \frac{10-0}{8} = 1.25$$

$$y_i = f(x_i) = \int_0^{10} \frac{300x}{1+e^x}\,\mathrm{dx}$$

i	x	y
0	$x_0 = a = 0$	$\frac{300 \times (0)}{1+e^{(0)}} = 0$
1	0+1.25=1.25	$\frac{300 \times (1.25)}{1+e^{(1.25)}} = 83.51$
2	1.25+1.25=2.5	$\frac{300 \times (2.5)}{1+e^{(2.5)}} = 56.89$
3	2.5+1.25=3.75	$\frac{300 \times (3.75)}{1+e^{(3.75)}} = 25.85$
4	3.75+1.25=5	$\frac{300 \times (5)}{1+e^{(5)}} = 10.04$
5	5+1.25=6.25	$\frac{300 \times (6.25)}{1+e^{(6.25)}} = 3.61$
6	6.25+1.25=7.5	$\frac{300 \times (7.5)}{1+e^{(7.5)}} = 1.24$
7	7.5+1.25=8.75	$\frac{300 \times (8.75)}{1+e^{(8.75)}} = 0.42$
8	8.75+1.25=10=b	$\frac{300 \times (10)}{1+e^{(10)}} = 0.14$

$$I = \frac{\Delta x}{2}[y_0 + y_n + \sum_{i=1}^{n-1} 2y_i]$$
$$= \frac{1.25}{2}[0 + 0.14 + 2 \times (83.51 + 56.89 + 25.85 + 10.04 + 3.61 + 1.24 + 0.42)] = 227.04$$

Compare with the true value:

$$\in = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{246.59 - 227.04}{246.59}\right| \times 100\% = 7.93\%$$

$$\left|\frac{11061 - 11266.4}{11061}\right| \times 100\% = 1.86\%$$

Ex3: Calculate the value of the given integration.

$$\int_{1}^{5} \sqrt{1+x^2}\,\mathrm{dx}$$

The number of segments (n) is equal to 8.

Solution:

$a = 1 \quad , \quad b = 5$

Eight segments (n=8)

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{8} = 0.5$$

$$y_i = f(x_i) = \sqrt{1+x^2}$$

i	x	y
0	$x_0 = a$ =1	$\sqrt{1+(1)^2} = 1.41$
1	1+0.5=1.5	$\sqrt{1+(1.5)^2} = 1.80$
2	1.5+0.5=2	$\sqrt{1+(2)^2} = 2.24$
3	2+0.5=2.5	$\sqrt{1+(2.5)^2} = 2.69$
4	2.5+0.5=3	$\sqrt{1+(3)^2} = 3.26$
5	3+0.5=3.5	$\sqrt{1+(3.5)^2} = 3.64$
6	3.5+0.5=4	$\sqrt{1+(4)^2} = 4.12$

7	4+0.5=4.5	$\sqrt{1+(4.5)^2}=4.61$
8	4.5+0.5=5=b	$\sqrt{1+(5)^2}=5.10$

$$I=\frac{\Delta x}{2}[y_0+y_n+\sum_{i=1}^{n-1}2y_i]$$
$$=\frac{0.5}{2}[1.41+5.10+2\times(1.80+2.24+2.69+3.26+3.64+4.12+4.61)]$$
$$=12.81$$

Homework 11

1. Use the trapezoidal method to estimate the value of the following integration.

$$\int_0^2 x^2 dx$$

The number of segments (n) is equal to 10.

Compare your solution with the exact solution.

2. Use the trapezoidal method to estimate the value of the following integration.

$$\int_0^4 xe^{2x} dx$$

The number of segments (n) is equal to 2 and 8.

Compare your solution with the exact solution ($I = 5216.92$).