University of Babylon
College of Engineering
Department of Environmental Engineering
Engineering Analysis I (ENAN 103)

Numerical Integration

Gaussian Quadrature Method

Undergraduate Level, 3th Stage

Mr. Waleed Ali Tameemi
Engineer/ College of Engineering/ Babylon University
M.Sc. Civil Engineering/ the University of Kansas/ USA

2016-2017

Lecture Outline

4.0 – Gaussian Quadrature Method

The integration of a function can be approximately calculated using Gaussian Quadrature Method as follows:

$$x = \frac{1}{2}(b + a) + \frac{1}{2}(b - a)t$$

$$dx = \frac{1}{2}(b - a)dt$$

$$\boldsymbol{I = \int_a^b f(x)dx = \int_{-1}^{1} f(t)dt = C_1 f(t_1) + C_2 f(t_2) + \ C_3 f(t_3)}$$

Then this can be solved as follows:

$$\boldsymbol{t_1 = \frac{-1}{\sqrt{3}}}$$

$$\boldsymbol{t_2 = 0}$$

$$\boldsymbol{t_3 = \frac{1}{\sqrt{3}}}$$

1. For two points quadrature,

$$\boldsymbol{C_1 = 1}$$

$$\boldsymbol{C_2 = 0}$$

$$\boldsymbol{C_3 = 1}$$

2. For three points quadrature,

$$\boldsymbol{C_1 = 0.555}$$

$$\boldsymbol{C_2 = 0.888}$$

$$\boldsymbol{C_3 = 0.555}$$

Ex1: Calculate the value of the given integration.

$$\int_0^{10} \frac{300x}{1+e^x}\,\mathrm{dx}$$

Use two points and three points quadrature.

Compare your solution with the exact solution (I $= 246.59$).

Solution:

$a = 0 \qquad\qquad b = 10$

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)t = \frac{1}{2}(10+0) + \frac{1}{2}(10-0)t = 5 + 5t$$

$$dx = \frac{1}{2}(b-a)dt = \frac{1}{2}(10-0)dt = 5dt$$

Sub. x and dx into $f(x)$

$$I = \int_0^{10} \frac{300x}{1+e^x}\,\mathrm{dx} = \int_{-1}^{1} \frac{300\times(5+5t)}{1+e^{(5+5t)}}\,5dt$$

Then this can be solved as follows:

$$I = \int_{-1}^{1} \frac{300\times(5+5t)}{1+e^{(5+5t)}}\,5dt = C_1 f(t_1) + C_2 f(t_2) + \; C_3 f(t_3)$$

$$t_1 = \frac{-1}{\sqrt{3}} \rightarrow f(t_1) = \frac{300\times(5+5\times(\frac{-1}{\sqrt{3}}))}{1+e^{(5+5\times(\frac{-1}{\sqrt{3}}))}} \times 5 = 341.8$$

$$t_2 = 0 \rightarrow f(t_2) = \frac{300\times(5+5\times(0))}{1+e^{(5+5\times(0))}} \times 5 = 50.2$$

$$t_3 = \frac{1}{\sqrt{3}} \rightarrow f(t_3) = \frac{300\times(5+5\times(\frac{1}{\sqrt{3}}))}{1+e^{(5+5\times(\frac{1}{\sqrt{3}}))}} \times 5 = 4.4$$

1. For two points quadrature,

   $C_1 = 1$

$C_2 = 0$

$C_3 = 1$

$$I = \int_{-1}^{1} \frac{300 \times (5 + 5t)}{1 + e^{(5+5t)}} 5dt = 1 \times (341.8) + 0 \times 50.2 + \ \ 1 \times 4.4 = 346.2$$

Compare with the true value:

$$\epsilon = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{246.59 - 346.2}{246.59}\right| \times 100\% = 40.4\%$$

2. For three points quadrature,

$C_1 = 0.555$

$C_2 = 0.888$

$C_3 = 0.555$

$$I = \int_{-1}^{1} \frac{300 \times (5 + 5t)}{1 + e^{(5+5t)}} 5dt = 0.555 \times (341.8) + 0.888 \times 50.2 + \ \ 0.555 \times 4.4 = 236.71$$

Compare with the true value:

$$\epsilon = \left|\frac{True\ Value - Approximate\ Value}{True\ Value}\right| \times 100\% = \left|\frac{246.59 - 236.71}{246.59}\right| \times 100\% = 4\%$$

Ex2: Calculate the value of the given integration.

$$\int_{0}^{1} \frac{x}{\sin x} \mathrm{dx}$$

Use two points and three points quadrature.

Solution:

$a = 0 \qquad b = 1$

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)t = \frac{1}{2}(1+0) + \frac{1}{2}(1-0)t = \frac{1}{2} + \frac{t}{2}$$

$$dx = \frac{1}{2}(b-a)dt = \frac{1}{2}(1-0)dt = \frac{dt}{2}$$

Sub. $x\ and\ dx$ into $f(x)$

$$I = \int_0^1 \frac{x}{\sin x}\,\mathrm{dx} = \int_{-1}^1 \frac{\frac{1}{2}+\frac{t}{2}}{\sin(\frac{1}{2}+\frac{t}{2})}\frac{dt}{2}$$

Then this can be solved as follows:

$$I = \int_{-1}^{1} \frac{\frac{1}{2}+\frac{t}{2}}{\sin(\frac{1}{2}+\frac{t}{2})}\frac{dt}{2} = C_1 f(t_1) + C_2 f(t_2) + \ C_3 f(t_3)$$

$$t_1 = \frac{-1}{\sqrt{3}} \to f(t_1) = \frac{\frac{1}{2}+\frac{(\frac{-1}{\sqrt{3}})}{2}}{\sin(\frac{1}{2}+\frac{(\frac{-1}{\sqrt{3}})}{2})}\frac{1}{2} = 0.50$$

$$t_2 = 0 \to f(t_2) = \frac{\frac{1}{2}+\frac{(0)}{2}}{\sin(\frac{1}{2}+\frac{(0)}{2})}\frac{1}{2} = 0.52$$

$$t_3 = \frac{1}{\sqrt{3}} =\to f(t_3) = \frac{\frac{1}{2}+\frac{(\frac{1}{\sqrt{3}})}{2}}{\sin(\frac{1}{2}+\frac{(\frac{1}{\sqrt{3}})}{2})}\frac{1}{2}0.56$$

3. For two points quadrature,

   $C_1 = 1$

   $C_2 = 0$

   $C_3 = 1$

$$I = \int_{-1}^{1} \frac{\frac{1}{2}+\frac{t}{2}}{\sin(\frac{1}{2}+\frac{t}{2})}\frac{dt}{2} = 1 \times 0.50 + 0 \times 0.52 + \ 1 \times 0.56 = 1.06$$

4. For three points quadrature,

   $C_1 = 0.555$

   $C_2 = 0.888$

   $C_3 = 0.555$

$$I = \int_{-1}^{1} \frac{\frac{1}{2}+\frac{t}{2}}{\sin(\frac{1}{2}+\frac{t}{2})}\frac{dt}{2} = 0.555 \times 0.50 + 0.888 \times 0.52 + \ 0.555 \times 0.56 = 1.05$$

Ex3: Calculate the value of the given integration.

$$\int_{1}^{5} \sqrt{1+x^2}\mathrm{dx}$$

Use two points and three points quadrature.

Solution:

$a = 1 \qquad\qquad b = 5$

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)t = \frac{1}{2}(1+5) + \frac{1}{2}(5-1)t = 3 + 2t$$

$$dx = \frac{1}{2}(b-a)dt = \frac{1}{2}(5-1)dt = 2dt$$

Sub. $x\ and\ dx$ into $f(x)$

$$I = \int_1^5 \sqrt{1 + x^2}\mathrm{dx} = \int_{-1}^1 \sqrt{1 + (3 + 2t)^2} \times 2\mathrm{dt}$$

Then this can be solved as follows:

$$I = \int_{-1}^{1} \sqrt{1 + (3 + 2t)^2} \times 2\mathrm{dt} = C_1 f(t_1) + C_2 f(t_2) + \ \ C_3 f(t_3)$$

$$t_1 = \frac{-1}{\sqrt{3}} \rightarrow f(t_1) = \sqrt{1 + (3 + 2(\frac{-1}{\sqrt{3}})^2} \times 2 = 4.2$$

$$t_2 = 0 \rightarrow f(t_2) = \sqrt{1 + (3 + 2(0))^2} \times 2 = 6.3$$

$$t_3 = \frac{1}{\sqrt{3}} \rightarrow f(t_3) = \sqrt{1 + (3 + 2(\frac{1}{\sqrt{3}}))^2} \times 2 = 8.5$$

1. For two points quadrature,

   $C_1 = 1$

   $C_2 = 0$

   $C_3 = 1$

$$I = \int_{-1}^{1} \sqrt{1 + (3 + 2t)^2} \times 2 = 1 \times 4.2 + 0 \times 6.3 + \ \ 1 \times 8.5 = 12.7$$

2. For three points quadrature,

   $C_1 = 0.555$

   $C_2 = 0.888$

   $C_3 = 0.555$

$$I = \int_{-1}^{1} \frac{300 \times (5 + 5t)}{1 + e^{(5+5t)}} 5t = 0.555 \times 4.2 + 0.888 \times 6.3 + \ \ 0.555 \times 8.5 = 12.6$$

Homework 15

Use Gaussian Quadrature method to estimate the value of the following integration.

$$\int_{0}^{4} xe^{2x}\,dx$$

Use two points and three points quadrature.

Compare your solution with the exact solution ($I = 5216.92$).