Counting Techniques

Counting Techniques

- Combinatorics is a fascinating branch of discrete mathematics, which deals with the art of counting. Very often we ask the question, In how many ways can a certain task be done? Usually combinatorics comes to our rescue. In most cases, listing the possibilities and counting them is the least desirable way of finding the answer to such a problem. Often we are not interested in enumerating the possibilities, but rather would like to know the total number of ways the task can be done.

Counting Techniques (Cont.)

Since in most cases it not feasible to list all the outcomes, we use the following techniques to COUNT them when listing them:

- Fundamental Counting Rule
- Permutations
- Combinations

Addition Principle

- Let A and B be two mutually exclusive tasks. Suppose task A can be done in m ways and task B in n ways. Then task A or task B can take place in m + n ways. ■

Example

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
- Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Example

A freshman has selected four courses and needs one more course for the next term. There are 15 courses in English, 10 in French, and 6 in German she is eligible to take. In how many ways can she choose the fifth course?

SOLUTION:

Let E be the task of selecting a course in English, F the task of selecting a course in French, and G that of selecting a course in German. These tasks can be done in 15, 10, and 6 ways, respectively, and are mutually exclusive, so, by the addition principle, the fifth course can be selected in

$$|E| + |F| + |G| = 15 + 10 + 6 = 31 \text{ ways.} \qquad \blacksquare$$

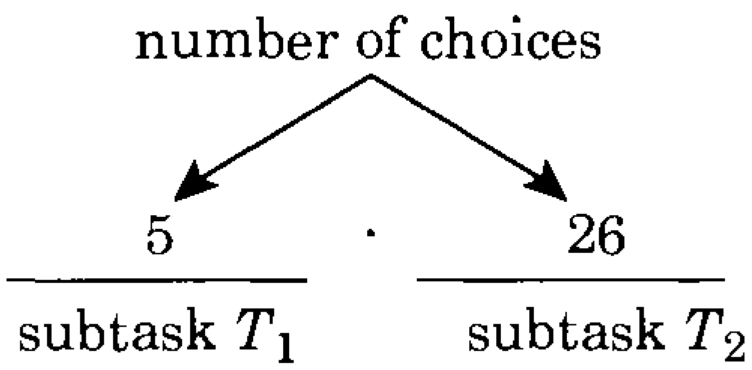
Multiplication Principle

- Suppose a task T is made up of two subtasks, subtask T_1 followed by sub task T_2. If subtask T_1 can be done in m_1 ways and subtask T_2 in m_2 different ways for each way subtask T_1 can be done, then task T can be done in $m_1 * m_2$ ways.

Example

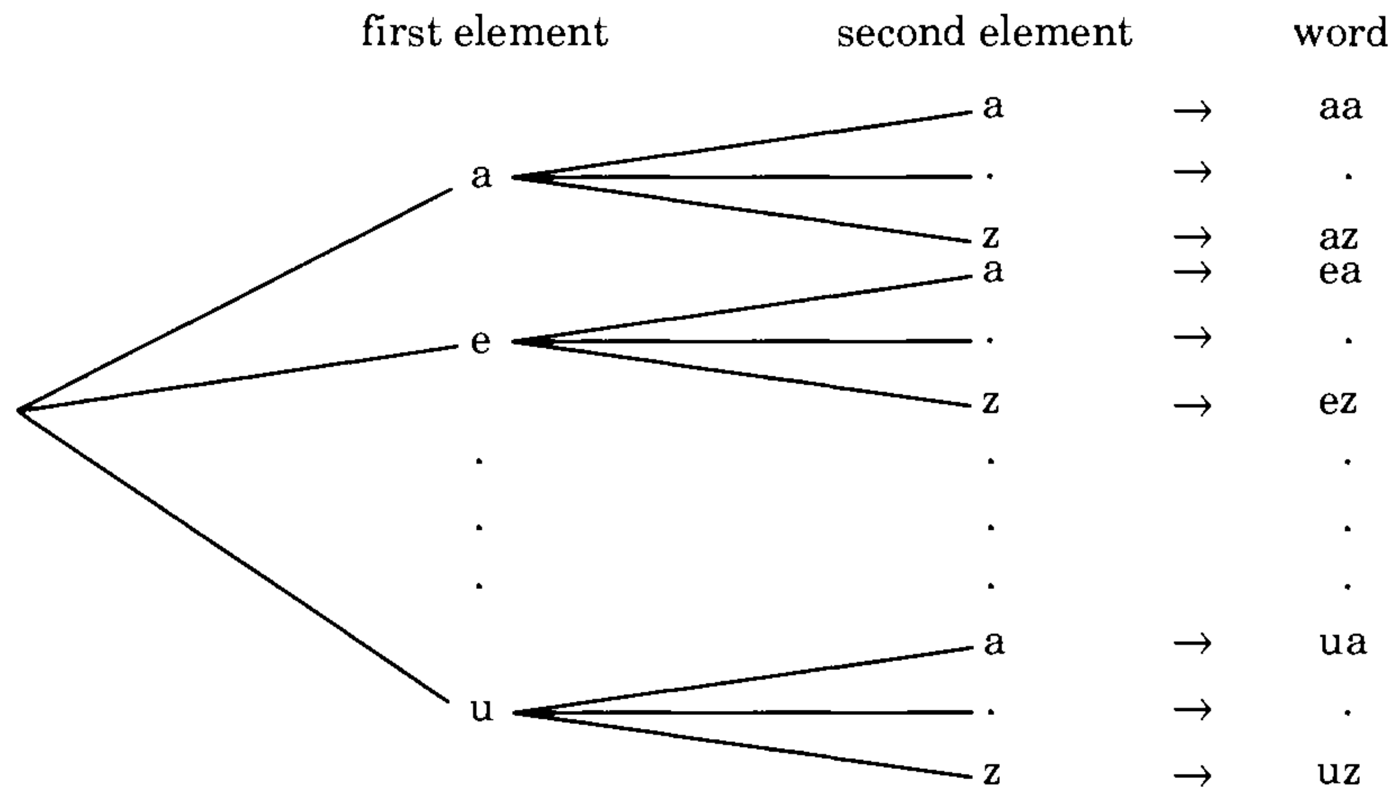
- Find the number of two-letter words that begin with a vowel — a, e, i, o, or u.
- SOLUTION:

The task of forming a two-letter word consists of two subtasks T1 and T2: T1 consists of selecting the first letter and T2 selecting the second letter, as in the following Figure.



Example (Cont.)

- The various two-letter words in this example can be enumerated systematically by constructing a tree diagram, as in the following Figure. All desired words can be obtained by traversing the various branches of the tree, as indicated.



Example

- A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
- Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

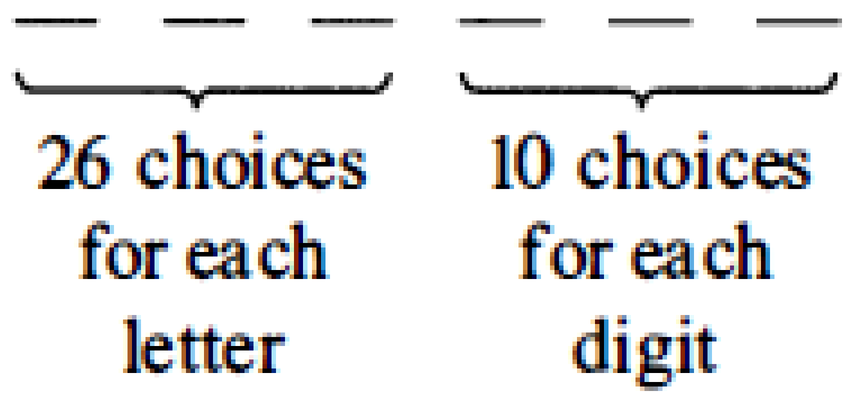
Example

- How many different bit strings of length seven are there?

- Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

Example

- How many different license plates are available if each plate contains a sequence of three letters followed by three digits?
- Solution: There are 26 choices for each of the three letters and ten choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.



Example

- Telephone number in the United States consists of 10 digits as in the following table. How many possible values for old & new number plans?

Term	Possible Values	Notes
X	0 – 9	10 values
N	2 – 9	8 values
Y	0 or 1	2 values
Old Number Plan	NYX-NNX-XXXX	area code, office code, and station code
New Number Plan	NXX-NXX-XXXX	area code, office code, and station code

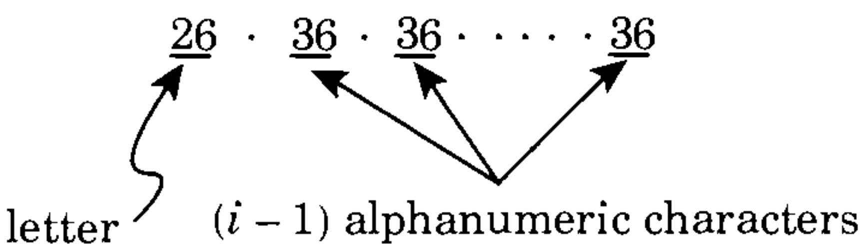
Example (Cont.), Solution

Term	Code	Possible values
Area code	NYX	8 . 2 . 10 = 160
	NXX	8 . 10 . 10 = 800
Office code	NNX	8 . 8 . 10 = 640
Station code	XXXX	10 . 10 . 10 . 10 = 10,000
Old Plan	NYX-NNX-XXXX	160 . 640 . 10,000 = 1,024,000,000
New Plan	NXX-NXX-XXXX	800 . 800 . 10,000 = 6,400,000,000

Example

- An identifier in a programming language consists of a letter followed by alphanumeric characters. Find the number of legal identifiers of length at most 10?

Let S_i denote the set of identifiers of length i, where $1 \leq i \leq 10$. Then $|S_i| = 26 \cdot 36^{i-1}$ Since the subtasks $S_1, \ldots, S_{10}$ are mutually



$$\sum_{i=1}^{10} |S_i| = \sum_{i=1}^{10} 26 \cdot 36^{i-1} = 26 \left(\sum_{i=0}^{9} 36^i \right)$$

$$= 26 \cdot \frac{(36^{10} - 1)}{36 - 1} = \frac{26(36^{10} - 1)}{35}$$

$$= 2{,}716{,}003{,}412{,}618{,}210$$

$$\approx 2.7 \text{ quadrillion!}$$

Example

- An eight-bit word is called a **byte**. Find the number of bytes with their second bit 0 or the third bit 1 ?

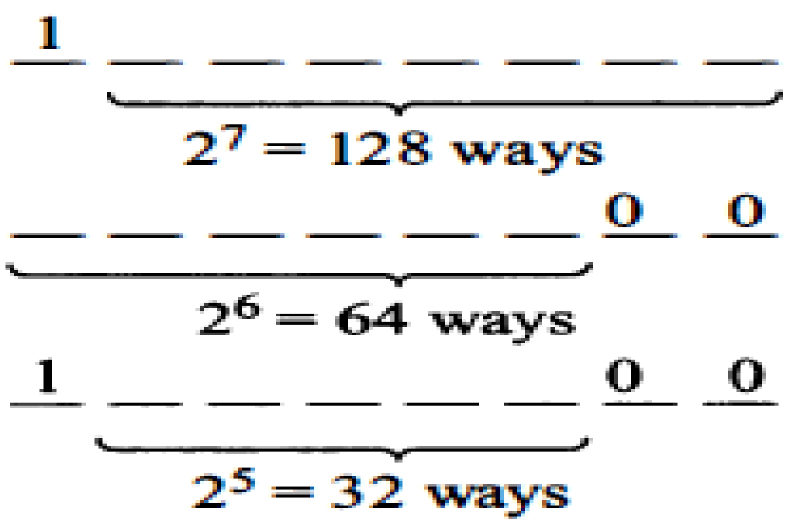
$$\text{Number of bytes with second bit } 0 = 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$$

$$\text{Number of bytes with third bit } 1 = 2 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$$

- The answer is $2^7 + 2^7 = 128 + 128 = 256$.

Example (Inclusion-Exclusion Principal)

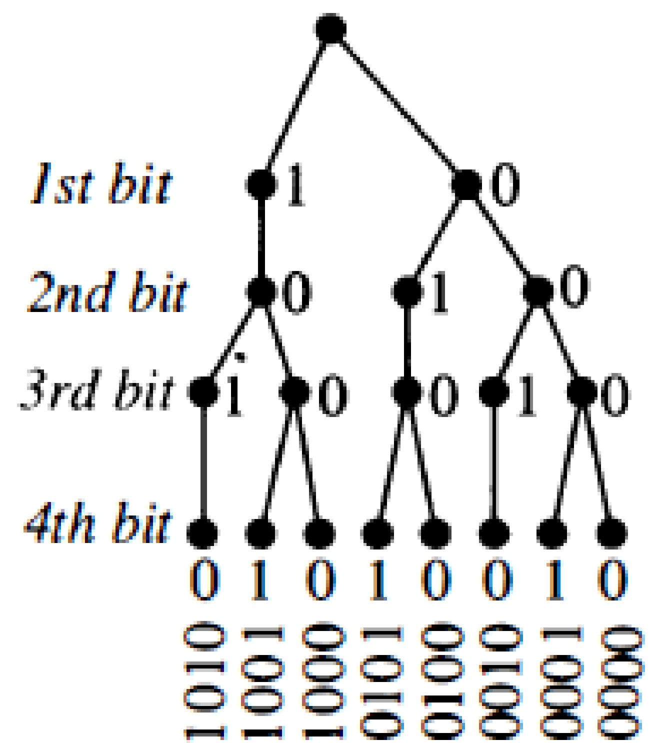
- How many bit strings of length eight either start with a 1 bit or end with the two bits 00?



- 128 + 64 - 32 = 160

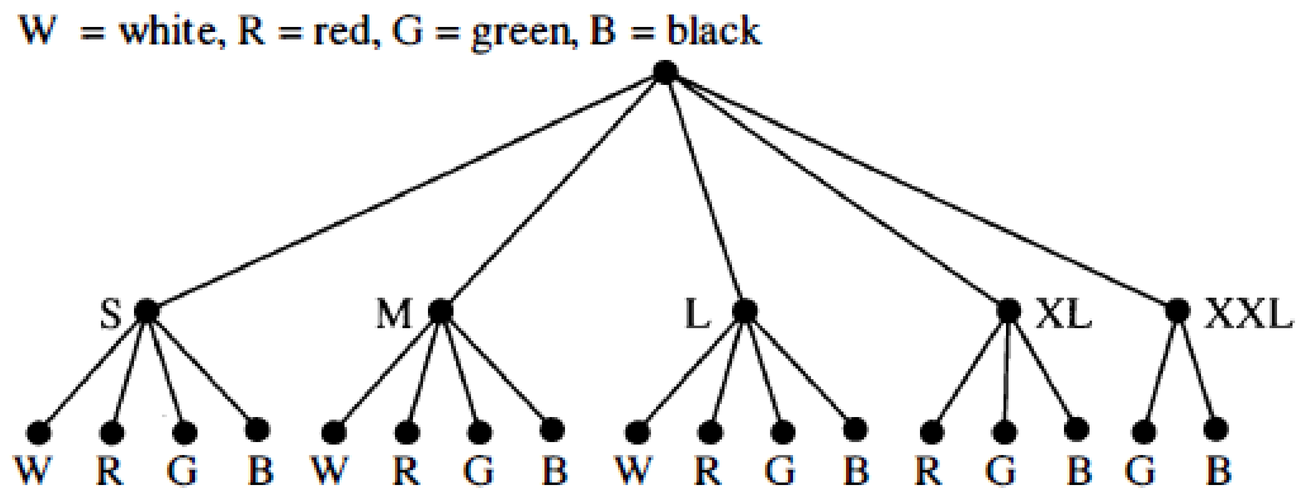
Tree Diagrams

- Counting problems can be solved using tree diagrams. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches.
- Example: How many bit strings of length four do not have two consecutive 1 s?



Example

- Suppose that "I Love Babylon" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?



Permutations

- When a group of objects or people are arranged in a certain order, this arrangement is called a permutation
- For permutations, the order is important (for combinations, order is unimportant.
- You can find the number of permutations by making use of the fundamental counting principal and multiplying the choices for each category together
- Or you can use the following formula:
- The number of permutations of n distinct objects taken r at a time is given by:
- $P(n,r) = n! / (n - r)!$
- !Can't use if choices may be repeated!

Example 1

Eight people enter the Best Pic contest. How many ways can blue, red, and green ribbons be awarded?

Since each winner will receive a different ribbon, order is important. You must find the number of permutations of 8 things taken 3 at a time.

$$P(n, r) = \frac{n!}{(n-r)!}$$ Permutation formula

$$= \frac{8!}{(8-3)!}$$ $n = 8, r = 3$

$$= \frac{8!}{5!}$$ Simplify.

Example 1

$$= \frac{8 \cdot 7 \cdot 6 \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{4}} \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{1}}}{\underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{4}} \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{1}}} \text{ or } 336$$

Divide by common factors.

Answer: The ribbons can be awarded in 336 ways.

Your Turn

Ten people are competing in a swim race where 4 ribbons will be given. How many ways can blue, red, green, and yellow ribbons be awarded?

Answer: 5040

Permutations with Repetitions

- Suppose I want to determine the number of ways I can arrange the letters in GEOMETRY
- If all the letters would be different we can find the arrangments using P(8,8) = 8! = 40,320.. But there is a problem!
- We have two identical E's… some of the 40,320 arrangements are identical except that the E's have been switched… which makes those arrangments INDISTINGUISHABLE PERMUTATIONS
- To account for the identical E's, divide P(8,8) by the number of arrangments of e
- The two e's can be arranged in P(2,2) or 2! Ways
- Thus the number of DISTINGUISHABLE permutations of the letters in GEOMETRY is 8! / 2! = 20,160
- For permutations with repeated elements, we use the rule below:
- The number of permutations of n objects of which p are alike and q are alike is (n! / p!q!)
- This rule can be extendend to any number of objects that are repeated!

Example 2

How many different ways can the letters of the word BANANA be arranged?

The second, fourth, and sixth letters are each A.

The third and fifth letters are each N.

You need to find the number of permutations of 6 letters of which 3 of one letter and 2 of another letter are the same.

$$\frac{6!}{3!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} \text{ or } 60$$

Answer: There are 60 ways to arrange the letters.

Your Turn

How many different ways can the letters of the word ALGEBRA be arranged?

Answer: 2520

Combinations!

- An arrangement or selection of objects in which the order is NOT important is called a combination
- The number of combinations of n objects taken r at a time is written C(n,r) or ${}_nC_r$
- We know that there are P(n,r) ways to select r objects from a group of n if the order is important
- For each selection of r objects, there are r! ways to order the selected objects, all of which are considered to represent the SAME combination
- For example, if I choose letters C,A,T from CATHY, I could arrange them as CAT, ACT, TAC, CTA, TCA, ATC.. But all of these are considered to be the same combination
- Therefore $C(n,r) = P(n,r) / r!$ or $(n! / (n-r)!r!)$

Example 3

Five cousins at a family reunion decide that three of them will go to pick up a pizza. How many ways can they choose three people to go?

Since the order they choose is not important, you must find the number of combinations of 5 cousins taken three at a time.

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Combination formula

$$= \frac{5!}{(5-3)!3!}$$

$n = 5$ and $r = 3$

Example 3

$$= \frac{5!}{2!3!} \text{ or } 10$$ Simplify.

Answer: There are 10 ways to choose three people from the five cousins.

Your Turn

Six friends at a party decide that three of them will go to pick up a movie. How many ways can they choose three people to go?

Answer: 20 ways

Multiple Events

- In more complicated situations, you can use the Fundamental Counting Principle in conjunction with permutations/combinations to determine the number of possibilities

- For example, suppose I have a class with 15 boys and 10 girls and I want to send 2 boys and 2 girls to represent our Algebra class at the U.N.
- Do C(15,2) for the boys and C(10,2) for the girls, then multiply these answers together!

Example 4

Six cards are drawn from a standard deck of cards. How many hands consist of two hearts and four spades?

By the Fundamental Counting Principle, you can multiply the number of ways to select two hearts and the number of ways to select four spades.

Only the cards in the hand matter, not the order in which they were drawn, so use combinations.

$C(13, 2)$ Two of 13 hearts are to be drawn.

$C(13, 4)$ Four of 13 spades are to be drawn.

Example 4

Combination formula

$$C(13,\ 2) \cdot C(13,\ 4) = \frac{13!}{(13-2)!2!} \cdot \frac{13!}{(13-4)!4!}$$

Subtract.

$$= \frac{13!}{11!2!} \cdot \frac{13!}{9!4!}$$

Simplify.

$$= 78 \cdot 715 \text{ or } 55{,}770$$

Answer: There are 55,770 hands consisting of 2 hearts and 4 spades.

Your Turn

Thirteen cards are drawn from a standard deck of cards. How many hands consist of six hearts and seven diamonds?

Answer: 2,944,656 hands

Binomial Coefficients

$$(x+y)^3 = (x+y)(x+y)(x+y) = (xx+xy+yx+yy)(x+y)$$
$$= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$= x^3 + 3x^2y + 3xy^2 + y^3.$$

(Binomial coefficient) Let $n, k \in \mathbb{N}$. The symbol $\binom{n}{k}$ denotes the number of k-element subsets of an n-element set.

- We call the number $\binom{n}{k}$ a binomial coefficient. The reason for this nomenclature is that the numbers $\binom{n}{k}$ are the coefficients of binomial $(x + y)^n$. This is explained more thoroughly below.

Binomial Coefficients (Cont.)

- The non-zero values of $\binom{5}{k}$ is as follows:

1 5 10 10 5 1

$$
\begin{aligned}
(x+y)^5 &= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.
\end{aligned}
$$

Binomial Coefficient (Cont.)

- Interesting relation

$$\binom{n}{n-r} = \binom{n}{r} \text{ or, in other words, if } a+b=n \text{ then } \binom{n}{a} = \binom{n}{b}$$

- Example

Compute $\binom{10}{7}$. By definition,

$$\binom{10}{7} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 120$$

On the other hand, $10 - 7 = 3$ and so we can also compute $\binom{10}{7}$ as follows:

$$\binom{10}{7} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$$

Observe that the second method saves space and time.

Pascal Triangle

- The various binomial coefficients $\binom{n}{k}$, where $0 \le k \le n$, can be arranged in the form of a triangle, called Pascal's triangle, as shown below.

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \leftarrow \text{row } 0 \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \leftarrow \text{row } 1 \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \leftarrow \text{row } 2 \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \leftarrow \text{row } 3 \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \quad \leftarrow \text{row } 4
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & & & & 1 & & & & & \leftarrow \text{row } 0 \\
 & & & 1 & & 1 & & & & \leftarrow \text{row } 1 \\
 & & 1 & & 2 & & 1 & & & \leftarrow \text{row } 2 \\
 & 1 & & 3 & \blacktriangledown & 3 & & 1 & & \leftarrow \text{row } 3 \\
1 & & 4 & & 6 & & 4 & & 1 & \leftarrow \text{row } 4
\end{array}
$$

Pascal Triangle (Cont.)

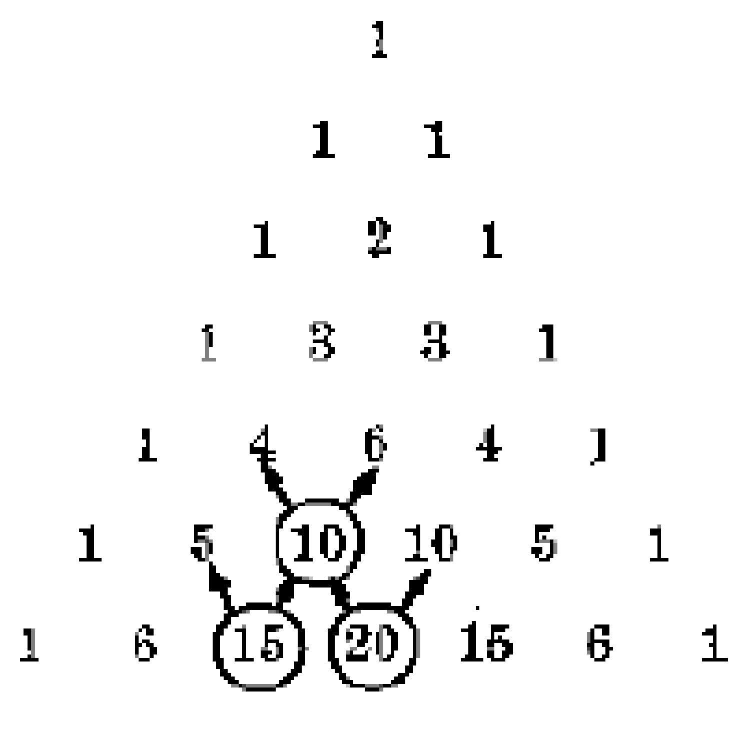
- The zeroth row of Pascal's triangle contains just the single number 1.
- Each successive row contains one more number than its predecessor.
- The first and last number in every row is 1.
- An intermediate number in any row is formed by adding the two numbers just to its left and just to its right in the previous row.

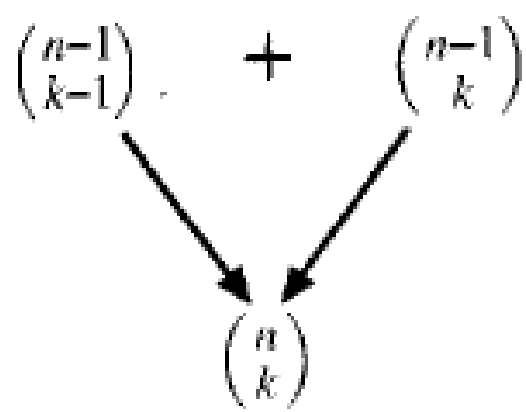
Pascal Triangle (Cont.)

- From the last property of Pascal Triangle,

(Pascal's Identity) Let n and k be integers with $0 < k < n$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$





Binomial Theorm

(The Binomial Theorem)* Let x and y be any real numbers, and n any nonnegative integer. Then $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$.

- Example

What is the expansion of $(x+y)^4$?

Solution: From the Binomial Theorem it follows that

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

The date for

the next exam is

January 4, 2012;

Wednesday.

Be ready for the exam on the next lecture

Thank You for Listening.