Elementary example

The simplest kind of linear system involves two equations and two variables:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 9y &= 15. \end{aligned}$$

One method for solving such a system is as follows. First, solve the top equation for x in terms of y:

$$x = 3 - \frac{3}{2}y.$$

Now substitute this expression for *x* into the bottom equation:

$$4\left(3 - \frac{3}{2}y\right) + 9y = 15.$$

This results in a single equation involving only the variable y. Solving gives $y = 1$, and substituting this back into the equation for x yields $x = 3/2$. This method generalizes to systems with additional variables (see "elimination of variables" below, or the article on elementary algebra.)

General form

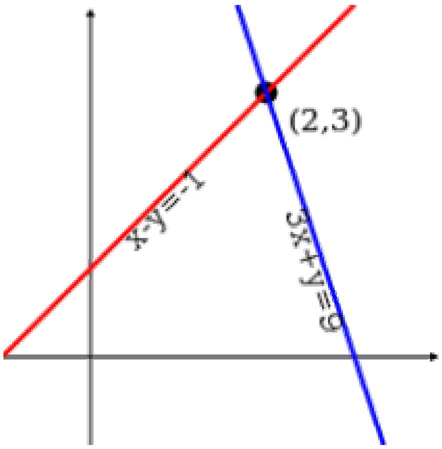
A general system of *m* linear equations with *n* unknowns can be written as

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m. \end{matrix}$$

Here $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

Often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen, as are polynomials and elements of an abstract algebraic structure.

Solution set



The solution set for the equations $x - y = -1$ and $3x + y = 9$ is the single point (2, 3).

A **solution** of a linear system is an assignment of values to the variables $x_1, x_2, ..., x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the **solution set**.

A linear system may behave in any one of three possible ways:

1. The system has *infinitely many solutions*.
2. The system has a single *unique solution*.
3. The system has *no solution*.

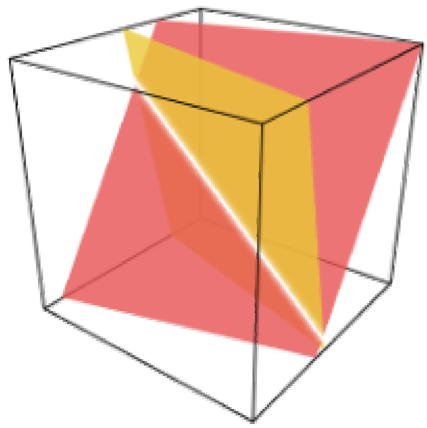
Geometric interpretation

For a system involving two variables (x and y), each linear equation determines a line on the xy-plane. Because a solution to a linear system must satisfy all of the equations, the solution set is the intersection of these lines, and is hence either a line, a single point, or the empty set.

For three variables, each linear equation determines a plane in three-dimensional space, and the solution set is the intersection of these planes. Thus the solution set may be a plane, a line, a single point, or the empty set. For example, as three parallel planes do not have a common point, the solution set of their equations is empty; the solution set of the equations of three planes intersecting at a point is single point; if three planes pass through two points, their equations have at least two common solutions; in fact the solution set is infinite and consists in all the line passing through these points.[2]

For n variables, each linear equation determines a hyperplane in n-dimensional space. The solution set is the intersection of these hyperplanes, which may be a flat of any dimension.

General behavior



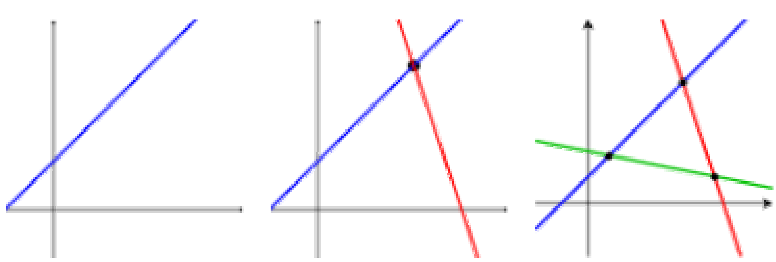
The solution set for two equations in three variables is usually a line.

In general, the behavior of a linear system is determined by the relationship between the number of equations and the number of unknowns:

- Usually, a system with fewer equations than unknowns has infinitely many solutions, but it may have no solution. Such a system is known as an underdetermined system.
- Usually, a system with the same number of equations and unknowns has a single unique solution.
- Usually, a system with more equations than unknowns has no solution. Such a system is also known as an overdetermined system.

In the first case, the dimension of the solution set is usually equal to $n - m$, where n is the number of variables and m is the number of equations.

The following pictures illustrate this trichotomy in the case of two variables:



One equation Two equations Three equations

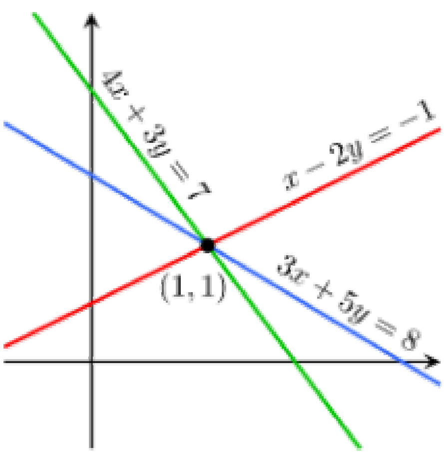
The first system has infinitely many solutions, namely all of the points on the blue line. The second system has a single unique solution, namely the intersection of the two lines. The third system has no solutions, since the three lines share no common point.

Keep in mind that the pictures above show only the most common case. It is possible for a system of two equations and two unknowns to have no solution (if the two lines are parallel), or for a system of three equations and two unknowns to be solvable (if the three lines intersect at a single point). In general, a system of linear equations may behave differently from expected if the equations are **linearly dependent**, or if two or more of the equations are **inconsistent**.

Properties

Independence

The equations of a linear system are **independent** if none of the equations can be derived algebraically from the others. When the equations are independent, each equation contains new information about the variables, and removing any of the equations increases the size of the solution set. For linear equations, logical independence is the same as linear independence.



The equations $x - 2y = -1$, $3x + 5y = 8$, and $4x + 3y = 7$ are linearly dependent.

For example, the equations

$$3x + 2y = 6 \quad \text{and} \quad 6x + 4y = 12$$

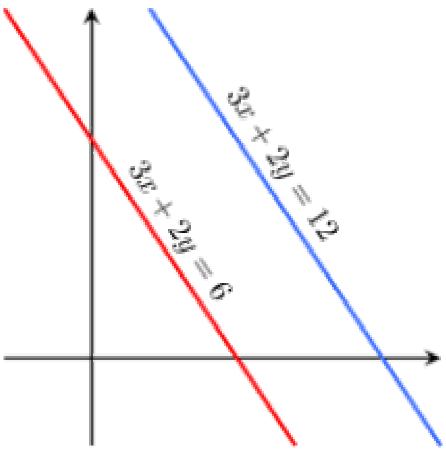
are not independent — they are the same equation when scaled by a factor of two, and they would produce identical graphs. This is an example of equivalence in a system of linear equations.

For a more complicated example, the equations

$$\begin{aligned} x - 2y &= -1 \\ 3x + 5y &= 8 \\ 4x + 3y &= 7 \end{aligned}$$

are not independent, because the third equation is the sum of the other two. Indeed, any one of these equations can be derived from the other two, and any one of the equations can be removed without affecting the solution set. The graphs of these equations are three lines that intersect at a single point.

Consistency



The equations 3*x* + 2*y* = 6 and 3*x* + 2*y* = 12 are inconsistent.

A linear system is **inconsistent** if it has no solution, and otherwise it is said to be **consistent** . When the system is inconsistent, it is possible to derive a contradiction from the equations, that may always be rewritten such as the statement 0 = 1.

For example, the equations

$$3x + 2y = 6 \quad \text{and} \quad 3x + 2y = 12$$

are inconsistent. In fact, by subtracting the first equation from the second one and multiplying both sides of the result by 1/6, we get 0 = 1. The graphs of these equations on the *xy*-plane are a pair of parallel lines.

It is possible for three linear equations to be inconsistent, even though any two of them are consistent together. For example, the equations

$$\begin{aligned} x + \quad y &= 1 \\ 2x + \quad y &= 1 \\ 3x + 2y &= 3 \end{aligned}$$

are inconsistent. Adding the first two equations together gives $3x + 2y = 2$, which can be subtracted from the third equation to yield $0 = 1$. Note that any two of these equations have a common solution. The same phenomenon can occur for any number of equations.

In general, inconsistencies occur if the left-hand sides of the equations in a system are linearly dependent, and the constant terms do not satisfy the dependence relation. A system of equations whose left-hand sides are linearly independent is always consistent.

Putting it another way, according to the Rouché–Capelli theorem, any system of equations (overdetermined or otherwise) is inconsistent if the rank of the augmented matrix is greater than the rank of the coefficient matrix. If, on the other hand, the ranks of these two matrices are equal, the system must have at least one solution. The solution is unique if and only if the rank equals the number of variables. Otherwise the general solution has k free parameters where k is the difference between the number of variables and the rank; hence in such a case there are an infinitude of solutions. The rank of a system of equations can never be higher than [the number of variables] + 1, which means that a system with any number of equations can always be reduced to a system that has a number of independent equations that is at most equal to [the number of variables] + 1.

Equivalence

Two linear systems using the same set of variables are **equivalent** if each of the equations in the second system can be derived algebraically from the equations in the first system, and vice-versa. Two systems are equivalent if either both are inconsistent or each equation of any of them is a linear combination of the equations of the other one. It follows that two linear systems are equivalent if and only if they have the same solution set.

Solving a linear system

There are several algorithms for solving a system of linear equations.

Describing the solution

When the solution set is finite, it is reduced to a single element. In this case, the unique solution is described by a sequence of equations whose left hand sides are the names of

the unknowns and right hand sides are the corresponding values, for example $(x = 3,\ y = -2,\ z = 6)$. When an order on the unknowns has been fixed, for example the alphabetical order the solution may be described as a vector of values, like $(3,\ -2,\ 6)$ for the previous example.

It can be difficult to describe a set with infinite solutions. Typically, some of the variables are designated as **free** (or **independent**, or as **parameters**), meaning that they are allowed to take any value, while the remaining variables are **dependent** on the values of the free variables.

For example, consider the following system:

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \end{aligned}$$

The solution set to this system can be described by the following equations:

$$x = -7z - 1 \quad \text{and} \quad y = 3z + 2.$$

Here *z* is the free variable, while *x* and *y* are dependent on *z*. Any point in the solution set can be obtained by first choosing a value for *z*, and then computing the corresponding values for *x* and *y*.

Each free variable gives the solution space one degree of freedom, the number of which is equal to the dimension of the solution set. For example, the solution set for the above equation is a line, since a point in the solution set can be chosen by specifying the value of the parameter *z*. An infinite solution of higher order may describe a plane, or higher-dimensional set.

Different choices for the free variables may lead to different descriptions of the same solution set. For example, the solution to the above equations can alternatively be described as follows:

$$y = -\frac{3}{7}x + \frac{11}{7} \quad \text{and} \quad z = -\frac{1}{7}x - \frac{1}{7}.$$

Here *x* is the free variable, and *y* and *z* are dependent.

Elimination of variables

The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

1. In the first equation, solve for one of the variables in terms of the others.
2. Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
3. Continue until you have reduced the system to a single linear equation.
4. Solve this equation, and then back-substitute until the entire solution is found.

For example, consider the following system:

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

Solving the first equation for x gives $x = 5 + 2z - 3y$, and plugging this into the second and third equation yields

$$\begin{aligned} -4y + 12z &= -8 \\ -2y + 7z &= -2 \end{aligned}$$

Solving the first of these equations for y yields $y = 2 + 3z$, and plugging this into the second equation yields $z = 2$. We now have:

$$\begin{aligned} x &= 5 + 2z - 3y \\ y &= 2 + 3z \\ z &= 2 \end{aligned}$$

Substituting $z = 2$ into the second equation gives $y = 8$, and substituting $z = 2$ and $y = 8$ into the first equation yields $x = -15$. Therefore, the solution set is the single point $(x, y, z) = (-15, 8, 2)$.