Quasilinear PDE

A first-order quasilinear PDE with two independent variables has the general form

$$f(x,y,u)u_x + g(x,y,u)u_y = h(x,y,u)$$

The system of ordinary differential equations are

$$\frac{dx}{f(x,y,u)} = \frac{dy}{g(x,y,u)} = \frac{du}{h(x,y,u)}$$

The following examples explain how to find the solution $u(x,t)$.

Example 1: Solve the initial value problem $u_t + xu_x = u$ with $u(x,0) = x^2$

Solution :

$$\frac{dt}{1} = \frac{dx}{x} = \frac{du}{u}$$

$$\frac{dt}{1} = \frac{dx}{x} \quad \Rightarrow \quad t + C_1 = \ln x$$

$$x = Ce^t \quad \Rightarrow C = xe^{-t}$$

$$\frac{dt}{1} = \frac{du}{u} \quad \Rightarrow \quad t + C_2 = \ln u$$

$$u = Ke^t$$

$$u(x,t) = g(xe^{-t})e^t$$

$$u(x,0) = x^2 \quad \Rightarrow \quad g(x) = x^2$$

$$u(x,t) = (xe^{-t})^2 e^t = x^2 e^{-2t} e^t = x^2 e^{-t}$$

Example 2: Solve the initial value problem $u_t + xu_x = -u^2$ with $u(x,0) = \sin x$

Solution :

$$\frac{dt}{1} = \frac{dx}{x} = \frac{du}{-u^2}$$

$$\frac{dt}{1} = \frac{dx}{x} \quad \Rightarrow \quad t + C_1 = \ln x$$

$$x = Ce^t \quad \Rightarrow C = xe^{-t}$$

$$\text{And} \quad \frac{dt}{1} = \frac{du}{-u^2}$$

$$t + K = \frac{1}{u}$$

$$u = \frac{1}{t + K}$$

$$u(x,t) = \frac{1}{t + g(xe^{-t})}$$

$$u(x,0) = \sin x \;\Rightarrow\; \frac{1}{g(x)} = \sin x \;\Rightarrow\; g(x) = \csc x$$

$$u(x,t) = \frac{1}{t + \csc(xe^{-t})}$$

Example 3: Solve the initial value problem $xu_t - 2xtu_x = 2tu$ with $u(x,0) = x^3$

Solution :

$$\frac{dt}{x} = \frac{dx}{-2xt} = \frac{du}{2tu}$$

$$\frac{dt}{x} = \frac{dx}{-2xt} \quad \Rightarrow \quad 2tdt = -dx$$

$$t^2 + x = C$$

$$\text{And} \quad \frac{dx}{-2xt} = \frac{du}{2tu} \quad \Rightarrow \quad \frac{dx}{-x} = \frac{du}{u}$$

$$-\ln x + \ln K = \ln u \;\Rightarrow\; u = \frac{K}{x}$$

$$\text{Then} \qquad u(x,t) = \frac{g(t^2 + x)}{x}$$

$$u(x,0) = x^3 \quad \Rightarrow \quad \frac{g(x)}{x} = x^3 \quad \Rightarrow \quad g(x) = x^4$$

$$u(x,t) = \frac{(t^2 + x)^4}{x}$$

Example 4: Solve the initial value problem $u_t + 4u_x = u^2$ with $u(x,0) = \dfrac{1}{1+x^2}$

Solution :

$$\frac{dt}{1} = \frac{dx}{4} = \frac{du}{u^2}$$

$$\frac{dt}{1} = \frac{dx}{4} \quad \Rightarrow \quad dx = 4dt$$

$$x = 4t + C \quad \Rightarrow \quad x - 4t = C$$

$$\text{And} \quad \frac{dt}{1} = \frac{du}{u^2} \quad \Rightarrow \quad t + K = -\frac{1}{u}$$

$$u = \frac{-1}{t+K}$$

$$\text{Then} \quad u(x,t) = \frac{-1}{t + g(x-4t)}$$

$$u(x,0) = \frac{1}{1+x^2} \quad \Rightarrow \quad \frac{-1}{g(x)} = \frac{1}{1+x^2}$$

$$g(x) = -(x^2+1)$$

$$u(x,t) = \frac{-1}{t - (x-4t)^2 - 1}$$

H.W: Solve the initial value problems

1. $u_t - 2u_x = u^2 \quad u(x,0) = \sin x \quad Ans: u(x,t) = \dfrac{-1}{t - \csc(x+2t)}$

2. $u_t + xu_x = \dfrac{x}{2u}$ with $u(x,0) = x \quad Ans: u(x,t) = \sqrt{x + x^2e^{-2t} - xe^{-t}}$