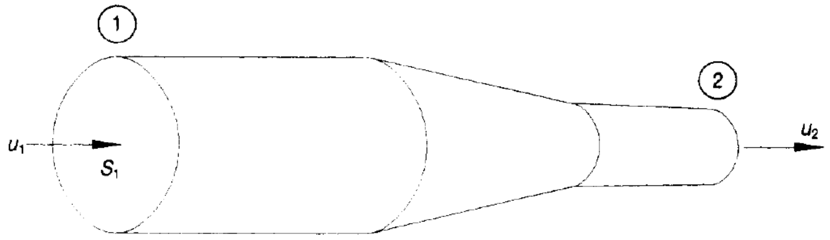
Conservation of mass

Consider flow through the pipe-work shown in Figure 1.3, in which the fluid occupies the whole cross section of the pipe. A mass balance can be written for the fixed section between planes 1 and 2, which are normal to the axis of the pipe. The mass flow rate across plane 1 into the section is equal to $\rho_1 Q_1$ and the mass flow rate across plane ***2*** out of the section is equal to $\rho_2 Q_2$**,** where $\boldsymbol{\rho}$ denotes the density of the fluid and Q the volumetric flow rate.

Thus, a mass balance can be written as

Mass flow rate in = mass flow rate out + rate of accumulation within section

$$\rho_1 Q_1 = \rho_2 Q_2 + \frac{\partial}{\partial t}(\rho_{av} V) \qquad \text{-------------------------------- } 1.3$$



$$\rho_1 Q_1 = \rho_2 Q_2 + V\frac{\partial \rho_{av}}{\partial t} \qquad \text{------------------------------- } 1.4$$

Where Vis the constant volume of the section between planes 1 and 2, and $\boldsymbol{\rho_{av}}$ is the density of the fluid averaged over the volume V. This equation represents the conservation of mass of the flowing fluid: it is frequently called the ‘continuity equation’ and the concept of ‘continuity’ is synonymous with the principle of conservation of mass.

In the case of unsteady compressible flow, the density of the fluid in the section will change and consequently the accumulation term will be non-zero. However, for steady compressible flow the time derivative must be zero by definition. In the case of incompressible flow, the density is constant so the time derivative is zero even if the flow is unsteady.

Thus, for incompressible flow or steady compressible flow, there is no accumulation within the section and consequently equation 1.4 reduces to

$$\rho_1 Q_1 = \rho_2 Q_2 \qquad \text{--------------------------------- } 1.5$$

This simply states that the mass flow rate into the section is equal to the mass flow rate out of the section. In general, the velocity of the fluid varies across the diameter of the pipe but an average velocity can be defined.

If the cross-sectional area of the pipe at a particular location is S, then the volumetric flow rate Q is given by

$$Q = uS \quad \text{------------------ } 1.6$$

Equation **1.6** defines the volumetric average velocity **u:** it is the uniform velocity required to give the volumetric flow rate Q through the flow area S. Substituting for **Q** in equation 1.5, the zero accumulation mass balance becomes

$$\rho_1 u_1 S_1 = \rho_2 u_2 S_2 \quad \text{---------------------- } 1.7$$

This is the form of the Continuity Equation that will be used most frequently but it is valid only when there is no accumulation. Although Figure 1.3 shows a pipe of circular cross section, equations 1.4 to 1.7 are valid for a cross section of any shape

problem

On a circular conduit there are different diameters: diameter $\mathbf{D_1 = 2\ m}$ changes into $\mathbf{D_2 = 3\ m}$. The velocity in the entrance profile was measured: $\mathbf{v_1 = 3\ ms^{-1}}$. Calculate the discharge and mean velocity at the outlet profile (see fig. 1). Determine also type of flow in both conduit profiles (whether the flow is laminar or turbulent) – temperature of water **T = 12° C**

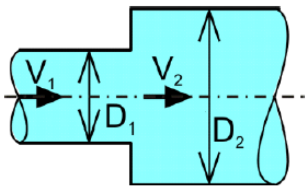


Figure 1

Solution

Discharge **Q** and consequently velocity $\mathbf{v_2}$ can be calculated from the continuity equation.

$$Q = v_1 \cdot S_1 = v_1 \cdot \frac{\pi \cdot D_1^2}{4} = 3 \cdot \frac{\pi \cdot 2^2}{4} = 9{,}425\,m^3s^{-1}$$

$$v_2 = \frac{Q}{S_2} = \frac{Q}{\pi \cdot D_2^2/4} = \frac{9{,}425}{\pi \cdot 3^2/4} = 1{,}333\,ms^{-1}$$

To determine type of flow in conduit, the Reynolds number $Re = \frac{v.D}{\nu}$ will be used.

For laminar flow: $Re < 2320$

For turbulent flow $Re > 2320$

Kinematic viscosity of water of 12°C: $\nu = 1{,}24 \cdot 10^{-6}\ m^2.\ s^{-1}$ (see Tab. 1)

For the conduit:

$$Re_1 = \frac{v_1 \cdot D_1}{\nu} = \frac{3.2}{1{,}24.10^{-6}} = 4838{,}1.10^3 > 2320 \ \text{turbulent flow}$$

$$Re_2 = \frac{v_2 \cdot D_2}{\nu} = \frac{1{,}333.3}{1{,}24.10^{-6}} = 3225.10^3 > 2320 \ \text{turbulent flow}$$

Energy relationships and the Bernoulli equation

The total energy of a fluid in motion consists of the following components: internal, potential, pressure and kinetic energies. Each of these energies may be considered with reference to an arbitrary base level. It is also convenient to make calculations on unit mass of fluid.

Potential energy this is the energy that a fluid has by virtue of its position in the Earth's field of gravity. The work required to raise a unit mass of fluid to a height ***z*** above an arbitrarily chosen datum is *zg*, where g is the acceleration due to gravity. This work is equal to the potential energy of unit mass of fluid above the datum.

Pressure energy this is the energy or work required to introduce the fluid into the system without a change of volume. If *P* is the pressure and V is the volume of mass ***m*** of fluid, then ***PV/***m is the pressure energy per unit mass of fluid. The ratio m /V is the fluid density ***ρ.*** Thus the pressure energy per unit mass of fluid is equal to ***P/ρ.***

Kinetic energy this is the energy of fluid motion. The kinetic energy of unit mass of the fluid is $v^2/2$, where ***v*** is the velocity of the fluid relative to some fixed body.

Total energy Summing these components, the total energy E per unit mass of fluid is given by the equation

$$E = U + zg + \frac{P}{\rho} + \frac{v^2}{2} \quad \text{---------------------------------} \quad 1.8$$

Consider fluid flowing from point 1 to point 2 as shown in Figure **below** Between these two points, let the following amounts of heat transfer and work is done per unit mass of fluid: heat transfer **q** to the fluid, work W_i done on the fluid and work W_o done by the fluid on its surroundings. W_i and W_o may be thought of as work input and output. Assuming the conditions to be steady, so that there is no accumulation of energy within the fluid between points 1 and 2, an energy balance can be written per unit mass of fluid as

$$E_1 + W_i + q = E_2 + W_o$$

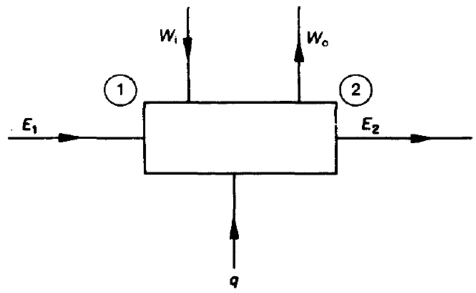
Or, after rearranging

$$E_2 = E_1 + q + W_i - W_o \quad \text{---} \quad 1.9$$

A flowing fluid is required to do work to overcome viscous frictional forces so that in practice the quantity W_o is always positive. It is zero only for the theoretical case of an inviscid fluid or ideal fluid having zero viscosity. The work W_i may be done on the fluid by a pump situated between points 1 and 2.

If the fluid has a constant density or behaves **as** an ideal gas, then the internal energy remains constant if the temperature is constant. If no heat transfer to the fluid takes place, ***q*** =0**.** For these conditions, equations 1.8 and 1.9 may be combined and written as

$$\left(z_2 g + \frac{P_2}{\rho_2} + \frac{v_2^2}{2}\right) = \left(z_1 g + \frac{P_1}{\rho_1} + \frac{v_1^2}{2}\right) + W_i - W_o \qquad (1.10)$$



For an inviscid fluid, ie frictionless flow, and no pump, equation (1.10) becomes

$$\left(z_2 g + \frac{P_2}{\rho_2} + \frac{v_2^2}{2}\right) = \left(z_1 g + \frac{P_1}{\rho_1} + \frac{v_1^2}{2}\right) \qquad (1.11)$$

Equation 1.11 is known as Bernoulli’s equation.

Dividing throughout by g, these equations can be written in a slightly different form. For example, equation 1.10 can be written as

$$\left(z_2 + \frac{P_2}{\rho_2 g} + \frac{v_2^2}{2g}\right) = \left(z_1 + \frac{P_1}{\rho_1 g} + \frac{v_1^2}{2g}\right) + \frac{W_i}{g} - \frac{W_o}{g} \qquad (1.12)$$

In this form, each term has the dimensions of length. The terms z, $P/(\rho g)$ and $\upsilon^2/(2g)$ are known as the potential, pressure and velocity heads, respectively. Denoting the work terms as heads, equation 1.12 can also be written as

$$\left(z_2 + \frac{P_2}{\rho_2 g} + \frac{v_2^2}{2g}\right) = \left(z_1 + \frac{P_1}{\rho_1 g} + \frac{v_1^2}{2g}\right) + \Delta h - h_f \qquad (1.13)$$

where Δh is the head imparted to the fluid by the pump and h_f is the head loss due to friction. The term Δh is known as the total head of the pump.

Equation 1.13 is simply an energy balance written for convenience in terms of length, ie heads. The various forms of the energy balance, equations 1.10 to 1.13, are often called Bernoulli’s equation

The various forms of energy are interchangeable and the equation enables these changes to be calculated in a given system. In deriving the form of Bernoulli’s equation without the work terms, it was assumed that the internal energy of the fluid remains constant. This is not the case when frictional dissipation occurs, ie there is a head loss ***hp*** In this case ***hf*** represents the conversion of mechanical energy into internal energy and, while internal energy can be recovered by heat transfer to a cooler medium, it cannot be converted into mechanical energy.

To enable Bernoulli’s equation to be used for the fluid flowing through the whole cross section of a pipe or duct, equation 1.13 can be modified as follows:

$$\left(z_2 + \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g\alpha}\right) = \left(z_1 + \frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g\alpha}\right) + \Delta h - h_f \qquad (1.14)$$

where u is the volumetric average velocity and α is a dimensionless correction factor, which accounts for the velocity distribution across the pipe or duct. For the relatively flat velocity profile that is found in turbulent flow, α has a value of approximately unity.

α has a value of $\frac{1}{2}$ for laminar flow of a Newtonian fluid in a pipe of circular section.

Reference